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Order-parameter flow in the SK spin-glass: II. Inclusion of microscopic memory effects

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Abstract. We develop further a recent dynamical replica theory to describe the dynamics of the Sherrington–Kirkpatrick spin-glass in terms of closed evolution equations for macroscopic order parameters. We show how microscopic memory effects can be included in the formalism through the introduction of a dynamic order parameter function: the joint spin-field distribution. The resulting formalism describes very accurately the relaxation phenomena observed in numerical simulations, including the typical overall slowing down of the flow that was missed by the previous simple two-parameter theory. The advanced dynamical replica theory is either exact or a very good approximation.

1. Introduction

In a previous paper [1] we introduced a theory to describe the Glauber dynamics of the Sherrington–Kirkpatrick (SK) [2] spin-glass model in terms of deterministic flow equations for two macroscopic state variables: the magnetization m and the spin-glass contribution r to the energy (for a more general discussion of the SK model and of the relevant literature on its dynamics we refer to [1]). The theory is based on the removal of microscopic memory effects: the only 'knowledge' the system is assumed to have of its past is the value of the macroscopic state (m, r). In fact any acceptable macroscopic dynamical theory for the SK model must contain as dynamical variables, either explicitly or implicitly, at least the magnetization m (which is the relevant observable for many typical spin-glass remanence phenomena) and the total energy of the system (in order for the theory to reproduce the correct equilibrium equations). Since for the SK model the energy per spin is a simple function of the macroscopic state vector (m, r), the theory of [1] can be seen as the simplest two-parameter dynamical theory for the SK model that has the properties of being exact for short times (upon choosing appropriate initial conditions) and in equilibrium.

At a technical level, the resulting formalism is a dynamical replica theory, which at fixed-points of the macroscopic flow reduces to the standard equilibrium replica theory, including replica symmetry breaking (RSB) à la Parisi [3] if it occurs. This is in contrast to an alternative formalism based on path-integral methods (see e.g. [4, 5]), where it is not yet known how to recover the standard equilibrium results in RSB situations. In fact, the potential of the present theory to provide the link between equilibrium replica theory and the description in terms of correlation and response functions (once the hitherto neglected

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microscopic memory effects have been incorporated), we regard as one of its most interesting features.

In [1] the actual macroscopic flow equations were determined explicitly only within the replica-symmetry (RS) ansatz. The RS version of the theory was quite successful at predicting the flow trajectories in the (m, r) plane, but also exhibited clear deviations in terms of the long-term temporal dependence of the macroscopic state variables m and r. These are partly due to the elimination from the dynamical equations of microscopic memory effects, and partly an artefact of the RS ansatz. The latter cause for deviations can be dealt with by allowing for a breaking of replica symmetry, following Parisi's RSB scheme [3]; although technically non-trivial, this can be seen as a straightforward generalization of the theory (dynamical RSB within the present formalism will be the subject of a subsequent paper [6]). Here, however, we concentrate on the more subtle question of how to incorporate more complete microscopic memory effects, i.e. on how to generalize the ideas in [1] to a situation where a macroscopic state specifies the details of the underlying microscopic states to a much higher degree. We will show that, by considering as the appropriate macroscopic dynamical observable the joint spin-field distribution, one can indeed follow the steps in [1] and arrive at a dynamic replica theory which not only inherits by construction the exactness of the previous (simple) two-parameter theory in the temporal limits $t \to 0$ and $t \to \infty$, but also describes the simulation experiments accurately, as far as our (limited) data allow us to conclude. The philosophy of our approach resembles the one proposed by Horner [7], who also derived a closed diffusion equation for the evolution of the joint spin-field distribution. However, at the technical level of the closure procedure there are important differences. The present theory is not only exact for short times (as is [7]), but also in equilibrium, in the sense that the *full* RSB equations are recovered. More importantly, it is constructed in such a way that it will produce exact dynamic equations if the joint spin-field distribution indeed turns out to constitute a closed level of description.

This paper is organized as follows. First we generalize the theory of [1] to the case of an arbitrary set of macroscopic observables, and derive constraints on the allowed choices for such a set, by requiring exactness in specific limits. We then derive a diffusion equation for the joint spin-field distribution, which generates a dynamical replica theory. For further explicit analysis, the relevant equations are simplified by making the RS ansatz, and their predictions are compared to the results of numerical simulations. We close the paper with a discussion of our results and their implications.

2. Dynamics and replicas

2.1. Definitions and macroscopic laws

The SK spin-glass model [2] consists of N Ising spins $\sigma_i \in \{-1, 1\}$ with infinite-range exchange interactions J_{ij}:

$$J_{ij} = \frac{1}{N} J_0 + \frac{1}{\sqrt{N}} J z_{ij} \qquad (i < j) \tag{1}$$

where the quantities z_{ij}, which represent quenched disorder, are drawn independently at random from a Gaussian distribution with zero mean and unit variance and then frozen during the spin dynamics. The evolution in time of the microscopic probability distribution $p_t(\boldsymbol{\sigma})$ is taken to be of the Glauber form described by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{\sigma}) = \sum_{k=1}^{N} [p_t(F_k\boldsymbol{\sigma}) w_k(F_k\boldsymbol{\sigma}) - p_t(\boldsymbol{\sigma}) w_k(\boldsymbol{\sigma})] \tag{2}$$

in which F_k is a spin-flip operator $F_k\Phi(\boldsymbol{\sigma}) \equiv \Phi(\sigma_1, \ldots, -\sigma_k, \ldots, \sigma_N)$ and the transition rates $w_k(\boldsymbol{\sigma})$ and the local alignment fields $h_i(\boldsymbol{\sigma})$ are

$$w_k(\boldsymbol{\sigma}) = \tfrac{1}{2}[1 - \sigma_k \tanh[\beta h_k(\boldsymbol{\sigma})]] \qquad h_i(\boldsymbol{\sigma}) = \sum_{j\neq i} J_{ij}\sigma_j + \theta \tag{3}$$

where β is the inverse temperature. This leads asymptotically to the required standard Boltzmann equilibrium distribution $p_\infty(\boldsymbol{\sigma}) \sim \exp[-\beta H(\boldsymbol{\sigma})]$, with the conventional Hamiltonian

$$H(\boldsymbol{\sigma}) = -\sum_{i<j} \sigma_i J_{ij} \sigma_j - \theta \sum_i \sigma_i. \tag{4}$$

We now turn to the evolution in time of any given set of ℓ macroscopic observables $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = (\Omega_1(\boldsymbol{\sigma}), \ldots, \Omega_\ell(\boldsymbol{\sigma}))$, described by the macroscopic probability distribution $\mathcal{P}_t(\boldsymbol{\Omega}) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]$. We insert the master equation (2), and expand the result in powers of the 'discrete derivatives' $\Delta_i^\mu(\boldsymbol{\sigma}) = \Omega_\mu(F_i\boldsymbol{\sigma}) - \Omega_\mu(\boldsymbol{\sigma})$, which gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_t(\boldsymbol{\Omega}) = &-\sum_{\mu=1}^{\ell} \frac{\partial}{\partial \Omega_\mu}\left\{\mathcal{P}_t(\boldsymbol{\Omega})\left\langle \sum_i w_i(\boldsymbol{\sigma})\Delta_i^\mu(\boldsymbol{\sigma})\right\rangle_{\boldsymbol{\Omega},t}\right\} \\ &+\frac{1}{2}\sum_{\mu\nu=1}^{\ell} \frac{\partial^2}{\partial\Omega_\mu\partial\Omega_\nu}\left\{\mathcal{P}_t(\boldsymbol{\Omega})\left\langle \sum_i w_i(\boldsymbol{\sigma})\Delta_i^\mu(\boldsymbol{\sigma})\Delta_i^\nu(\boldsymbol{\sigma})\right\rangle_{\boldsymbol{\Omega},t}\right\} + \mathcal{O}(N\ell^3\Delta^3)\end{aligned} \tag{5}$$

where we introduced the sub-shell, or conditional, average

$$\langle f(\boldsymbol{\sigma})\rangle_{\boldsymbol{\Omega},t} = \frac{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]f(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]}.$$

If the second (diffusion) term, which is $\mathcal{O}(N\ell^2\Delta^2)$, vanishes for $N \to \infty$, equation (5) acquires the Liouville form, the solution of which describes the deterministic flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega}_t = \left\langle \sum_i w_i(\boldsymbol{\sigma})[\boldsymbol{\Omega}(F_i\boldsymbol{\sigma}) - \boldsymbol{\Omega}(\boldsymbol{\sigma})]\right\rangle_{\boldsymbol{\Omega}_t,t}. \tag{6}$$

Although exact for $N \to \infty$ (provided the diffusion term indeed vanishes), the set (6) need not be closed, due to the appearance of $p_t(\boldsymbol{\sigma})$ in the sub-shell average.

There are two *natural* ways for the set (6) to close. First, by the argument of the subshell average in (6) depending on $\boldsymbol{\sigma}$ only through $\boldsymbol{\Omega}(\boldsymbol{\sigma})$ (now $p_t(\boldsymbol{\sigma})$ will simply drop out), and second by the microscopic dynamics (2) allowing for equipartitioning solutions (where $p_t(\boldsymbol{\sigma})$ depends on $\boldsymbol{\sigma}$ only through $\boldsymbol{\Omega}(\boldsymbol{\sigma})$). In both cases one obtains the correct equations for $\boldsymbol{\Omega}_t$ upon simply eliminating $p_t(\boldsymbol{\sigma})$ from (6).

2.2. Closed flow equation for an order parameter function

Generalizing [1] to the present case, we now make the following assumptions:

(i) The observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$ are self-averaging with respect to the microscopic realization of the disorder $\{z_{ij}\}$, at any time.

(ii) In evaluating the sub-shell averages we assume equipartitioning of probability within the $\boldsymbol{\Omega}$-subshell of the ensemble.

As a result $p_t(\boldsymbol{\sigma})$ drops out, and the macroscopic equations (6) are replaced by closed ones, from which the unpleasant fraction is removed as in [1] using the replica identity

$$\frac{\sum_{\boldsymbol{\sigma}} \Phi(\boldsymbol{\sigma})W(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma})} = \lim_{n\to 0}\sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n}\Phi(\boldsymbol{\sigma}^1)\prod_{\alpha=1}^{n} W(\boldsymbol{\sigma}^\alpha). \tag{7}$$

We now obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega}_t = \left\langle \frac{\sum_{\boldsymbol{\sigma}} \delta[\boldsymbol{\Omega}-\boldsymbol{\Omega}(\boldsymbol{\sigma})]\sum_i w_i(\boldsymbol{\sigma})[\boldsymbol{\Omega}(F_i\boldsymbol{\sigma})-\boldsymbol{\Omega}(\boldsymbol{\sigma})]}{\sum_{\boldsymbol{\sigma}} \delta[\boldsymbol{\Omega}-\boldsymbol{\Omega}(\boldsymbol{\sigma})]} \right\rangle_{\{z_{ij}\}}$$

$$= \lim_{n\to 0}\sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n}\left\langle \sum_i w_i(\boldsymbol{\sigma})[\boldsymbol{\Omega}(F_i\boldsymbol{\sigma}^1)-\boldsymbol{\Omega}(\boldsymbol{\sigma}^1)]\prod_{\alpha=1}^{n}\delta[\boldsymbol{\Omega}-\boldsymbol{\Omega}(\boldsymbol{\sigma}^\alpha)]\right\rangle_{\{z_{ij}\}}. \qquad (8)$$

Our second assumption (equipartitioning) is the dangerous one; its impact on the accuracy of the theory depends critically on the choice made for the observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$. If, however, the observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$ indeed obey closed self-averaging dynamic equations, our closure procedure will be exact (see our reasoning above). Requiring the theory to be exact in two solvable limits (equilibrium and $J\to 0$, respectively) imposes constraints on the allowed choices for $\boldsymbol{\Omega}(\boldsymbol{\sigma})$. Since in equilibrium we have equipartitioning of probability in the energy-subshells (with the Hamiltonian (4)), and since for $J\to 0$ one obtains closed dynamic equations for the magnetization, our two requirements imply $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = (m(\boldsymbol{\sigma}), H(\boldsymbol{\sigma}), \ldots)$ (modulo equivalent combinations). For the SK model the energy per spin can be written as

$$H(\boldsymbol{\sigma})/N = -\tfrac{1}{2}J_0 m^2(\boldsymbol{\sigma}) - \theta m(\boldsymbol{\sigma}) - Jr(\boldsymbol{\sigma}) + \tfrac{1}{2}J_0/N \qquad (9)$$

with

$$m(\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \sigma_i \qquad r(\boldsymbol{\sigma}) = \frac{1}{N\sqrt{N}}\sum_{i<j}\sigma_i z_{ij}\sigma_j \qquad (10)$$

so the choice made in [1] leads to the simplest two-parameter theory that meets our requirements of exactness in the two solvable limits. Improving upon [1] implies including microscopic information beyond (m, r), i.e. adding observables to the set $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = (m(\boldsymbol{\sigma}), r(\boldsymbol{\sigma}))$. Addition of any finite number of observables, although technically simple, is not expected to give more than just minor corrections. In contrast we choose for the set of observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$ the (infinite dimensional) joint spin-field distribution:

$$D(\varsigma, h; \boldsymbol{\sigma}) = \frac{1}{N}\sum_i \delta_{\varsigma,\sigma_i}\delta[h - h_i(\boldsymbol{\sigma})] \qquad (11)$$

with the local fields (3). Our motivation for this choice is the following

(i) The previous two dynamic parameters $m(\boldsymbol{\sigma})$ and $r(\boldsymbol{\sigma})$ can be written as integrals over $D(\varsigma, h; \boldsymbol{\sigma})$, so the advanced theory automatically inherits the exactness in the two solvable limits $t\to\infty$ and $J\to 0$.

(ii) The order parameter function $D(\varsigma, h)$ specifies the underlying states $\boldsymbol{\sigma}$ to a much higher degree than (m, r); i.e. microscopic memory is taken into account.

(iii) The microscopic equation (2) itself is formulated in terms of spins and fields.

(iv) The choice (11) allows for immediate generalization to models without detailed balance and to soft-spin models.

To avoid all kinds of technical difficulties we assume that the distribution (11) is sufficiently well behaved; we assume that we can evaluate $D_t(\varsigma, h)$ in a number ℓ of field arguments h_μ and take the limit $\ell\to\infty$ *after* the limit $N\to\infty$. We then have 2ℓ observables $\Omega_{\varsigma\mu}(\boldsymbol{\sigma}) = D(\varsigma, h_\mu; \boldsymbol{\sigma})$, with $\mu = 1, \ldots, \ell$ and $\varsigma = \pm 1$. In order to work out equation (8) we calculate the discrete derivatives $\Delta_i^{\varsigma\mu}(\boldsymbol{\sigma}) = D(\varsigma, h_\mu; F_i\boldsymbol{\sigma}) - D(\varsigma, h_\mu; \boldsymbol{\sigma})$:

$$\Delta_i^{\varsigma\mu}(\boldsymbol{\sigma}) = \frac{2\sigma_i}{N\sqrt{N}}\sum_{j\neq i}\delta_{\varsigma,\sigma_j}\delta'[h_\mu - h_j(\boldsymbol{\sigma})]\left[\frac{J_0}{\sqrt{N}} + Jz_{ij}\right]$$

$$+\frac{2J^2}{N^2}\sum_{j\neq i}\delta_{\varsigma,\sigma_j}\delta''[h_\mu - h_j(\boldsymbol{\sigma})]z_{ij}^2 - \frac{1}{N}\varsigma\sigma_i\delta[h_\mu - h_i(\boldsymbol{\sigma})] + \mathcal{O}(N^{-\frac{3}{2}}) \qquad (12)$$

where primes indicate derivatives (in a distributional sense). Since $\Delta_i^{\varsigma\mu}(\boldsymbol{\sigma}) = \mathcal{O}(N^{-\frac{1}{2}})$, the diffusion term in (5) could be $\mathcal{O}(1)$. Explicit calculation, however, will show that it vanishes as $N^{-\frac{1}{2}}$; at this stage we anticipate that calculation and assume deterministic evolution. To suppress notation we write

$$\sum_{\sigma} \int \mathrm{d}H\, f(\sigma, H) D(\sigma, H) = \langle f(\sigma, H) \rangle_D.$$

In order to see clearly for which terms our two closure assumptions will actually be operational, we first work out the exact equation (6). We insert (12) into (6) and retain only the leading $\mathcal{O}(1)$ terms:

$$\begin{aligned}\frac{\partial}{\partial t} D_t(\varsigma, h) = {} & \tfrac{1}{2}[1 + \varsigma \tanh(\beta h)] D_t(-\varsigma, h) - \tfrac{1}{2}[1 - \varsigma \tanh(\beta h)] D_t(\varsigma, h) \\ & + \frac{\partial}{\partial h} \Bigg\{ D_t(\varsigma, h)[h - \theta - J_0 \langle \tanh(\beta H) \rangle_{D_t}] \\ & - \frac{J}{N\sqrt{N}} \sum_{i \neq j} z_{ij} \langle \tanh(\beta h_i(\boldsymbol{\sigma})) \delta_{\varsigma,\sigma_j} \delta[h - h_j(\boldsymbol{\sigma})] \rangle_{D_t;t} \Bigg\} \\ & + J^2 \frac{\partial^2}{\partial h^2} \Bigg\{ \frac{1}{N^2} \sum_{i \neq j} z_{ij}^2 \langle [1 - \sigma_i \tanh(\beta h_i(\boldsymbol{\sigma}))] \delta_{\varsigma,\sigma_j} \delta[h - h_j(\boldsymbol{\sigma})] \rangle_{D_t;t} \Bigg\} \\ & + \mathcal{O}(N^{-\frac{1}{2}}) \end{aligned} \tag{13}$$

with the sub-shell average

$$\langle f(\boldsymbol{\sigma}) \rangle_{D;t} = \frac{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) f(\boldsymbol{\sigma}) \prod_{\varsigma\mu} \delta[D(\varsigma, h_\mu) - \frac{1}{N} \sum_j \delta_{\varsigma,\sigma_j} \delta[h_\mu - h_j(\boldsymbol{\sigma})]]}{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \prod_{\varsigma\mu} \delta[D(\varsigma, h_\mu) - \frac{1}{N} \sum_j \delta_{\varsigma,\sigma_j} \delta[h_\mu - h_j(\boldsymbol{\sigma})]]}. \tag{14}$$

The *closed* dynamic equation (8) is subsequently obtained from (13) by elimination of $p_t(\boldsymbol{\sigma})$ and averaging over the disorder (using identity (7)):

$$\begin{aligned}\frac{\partial}{\partial t} D_t(\varsigma, h) = {} & \tfrac{1}{2}[1 + \varsigma \tanh(\beta h)] D_t(-\varsigma, h) - \tfrac{1}{2}[1 - \varsigma \tanh(\beta h)] D_t(\varsigma, h) \\ & + \frac{\partial}{\partial h} \Bigg\{ D_t(\varsigma, h)[h - \theta - J_0 \langle \tanh(\beta H) \rangle_{D_t}] \\ & - \frac{J}{N\sqrt{N}} \sum_{i \neq j} \langle\langle z_{ij} \tanh(\beta h_i(\boldsymbol{\sigma})) \delta_{\varsigma,\sigma_j} \delta[h - h_j(\boldsymbol{\sigma})] \rangle\rangle_{D_t} \Bigg\} \\ & + J^2 \frac{\partial^2}{\partial h^2} \Bigg\{ \frac{1}{N^2} \sum_{i \neq j} \langle\langle z_{ij}^2 [1 - \sigma_i \tanh(\beta h_i(\boldsymbol{\sigma}))] \delta_{\varsigma,\sigma_j} \delta[h - h_j(\boldsymbol{\sigma})] \rangle\rangle_{D_t} \Bigg\} \\ & + \mathcal{O}(N^{-\frac{1}{2}}) \end{aligned} \tag{15}$$

with

$$\begin{aligned}\langle\langle f[\boldsymbol{\sigma}; \{z_{kl}\}] \rangle\rangle_D = {} & \lim_{n \to 0} \sum_{\boldsymbol{\sigma}^1} \cdots \\ & \cdots \sum_{\boldsymbol{\sigma}^n} \left\langle f[\boldsymbol{\sigma}^1; \{z_{kl}\}] \prod_{\alpha=1}^{n} \prod_{\varsigma\mu} \delta\left[D(\varsigma, h_\mu) - \frac{1}{N} \sum_j \delta_{\varsigma,\sigma_j^\alpha} \delta[h_\mu - h_j(\boldsymbol{\sigma}^\alpha)] \right] \right\rangle_{\{z_{kl}\}}. \end{aligned} \tag{16}$$

Finally, it will become clear shortly that the diffusion term in (15) is relatively simple, essentially obtained by replacing $z_{ij}^2 \to 1$. As a result all complications are concentrated in a single term $\mathcal{A}$, and we find for $N \to \infty$ the relatively simple final expression

$$\frac{\partial}{\partial t} D_t(\varsigma, h) = \frac{1}{2}[1 + \varsigma \tanh(\beta h)] D_t(-\varsigma, h) - \frac{1}{2}[1 - \varsigma \tanh(\beta h)] D_t(\varsigma, h) + \frac{\partial}{\partial h}\Big\{ D_t(\varsigma, h)[h - \theta - J_0 \langle \tanh(\beta H)\rangle_{D_t} + \mathcal{A}[\varsigma, h; D_t]] + J^2[1 - \langle \sigma \tanh(\beta H)\rangle_{D_t}] \frac{\partial}{\partial h} D_t(\varsigma, h)\Big\} \quad (17)$$

with

$$\mathcal{A}[\varsigma, h; D_t] = -\lim_{N\to\infty} \frac{J}{N\sqrt{N}} \sum_{i\neq j} \langle\langle z_{ij} \tanh(\beta h_i(\boldsymbol{\sigma})) \delta_{\varsigma,\sigma_j} \delta[h - h_j(\boldsymbol{\sigma})]\rangle\rangle_{D_t}. \quad (18)$$

3. Replica calculation of the flow

We now turn to the evaluation of $\langle\langle f[\boldsymbol{\sigma}; \{z_{kl}]\rangle\rangle_D$.

3.1. Disorder and spin averages

In the familiar fashion for replica calculations we carry out the disorder averages before the spin averages. We remove the disorder dependence through the local fields from within the constraining delta functions, by inserting

$$1 = \prod_\alpha \prod_k \int \mathrm{d}H_k^\alpha\, \delta[H_k^\alpha - h_k(\boldsymbol{\sigma}^\alpha)] = \prod_\alpha \prod_k \int \frac{\mathrm{d}\hat{h}_k^\alpha\, \mathrm{d}H_k^\alpha}{2\pi} \exp[\mathrm{i}\hat{h}_k^\alpha (H_k^\alpha - h_k(\boldsymbol{\sigma}^\alpha))]$$

so that we can write (16) as

$$\langle\langle f[\boldsymbol{\sigma}; \{z_{kl}\}]\rangle\rangle_D = \lim_{n\to 0} \int [\mathrm{d}\boldsymbol{H}^1\, \mathrm{d}\hat{\boldsymbol{h}}^1] \cdots [\mathrm{d}\boldsymbol{H}^n\, \mathrm{d}\hat{\boldsymbol{h}}^n] \times \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \exp\left[\mathrm{i} \sum_i \sum_\alpha \hat{h}_i^\alpha [H_i^\alpha - \theta] - \frac{\mathrm{i}J_0}{N} \sum_{i\neq j} \sum_\alpha \hat{h}_i^\alpha \sigma_j^\alpha\right] \times \prod_{\varsigma\mu\alpha} \delta\left[D(\varsigma, h_\mu) - \frac{1}{N}\sum_j \delta_{\varsigma,\sigma_j^\alpha} \delta[h_\mu - H_j^\alpha]\right] \times \left\langle f[\boldsymbol{\sigma}^1; \{z_{kl}\}] \exp\left[-\frac{\mathrm{i}J}{\sqrt{N}} \sum_{i\neq j} z_{ij} \sum_\alpha \hat{h}_i^\alpha \sigma_j^\alpha\right]\right\rangle_{\{z_{kl}\}}. \quad (19)$$

After symmetrization due to $z_{ij} = z_{ji}$ and after using permutation symmetry with respect to site labels, we find that the disorder averages in (15) involve the following two integrals, encountered in the flow term (18), and in the diffusion term, respectively:

$$\int \prod_{i<j} Dz_{ij} \exp\left[-\frac{\mathrm{i}J}{\sqrt{N}} z_{ij} \sum_\alpha (\hat{h}_i^\alpha \sigma_j^\alpha + \hat{h}_j^\alpha \sigma_i^\alpha)\right] \left\{ \begin{matrix} \sqrt{N} z_{12} \\ z_{12}^2 \end{matrix} \right\} = \left\{ \begin{matrix} -\mathrm{i}J \sum_\alpha [\hat{h}_2^\alpha \sigma_1^\alpha + \hat{h}_1^\alpha \sigma_2^\alpha] \\ 1 \end{matrix} \right\} \prod_{i<j} \exp\left[-\frac{J^2}{2N}\left[\sum_\alpha (\hat{h}_i^\alpha \sigma_j^\alpha + \hat{h}_j^\alpha \sigma_i^\alpha)\right]^2\right] \quad (20)$$

with the Gaussian measure $Dz = (2\pi)^{-1/2} \mathrm{e}^{-z^2/2}$, and where we only retained the leading $\mathcal{O}(1)$ contributions. Applying (19) to the trivial function $f[\boldsymbol{\sigma}; \{z_{kl}\}] = 1$ gives a normalization relation, which we can use to avoid having to perform the remaining integrals. As a result we immediately find the diffusion term in (15) to be simply

$$J^2 \frac{\partial^2}{\partial h^2}\{D_t(\varsigma,h)[1-\langle\sigma\tanh(\beta H)\rangle_{D_t}]\} \tag{21}$$

which proves (17), whereas the remaining disorder-induced flow term (18) remains non-trivial. A similar calculation shows, upon substitution of (12) into (5), that the second term in the macroscopic stochastic equation (5) is of order $\ell^2 N^{-1}$. Given our assumption that the limit $\ell\to\infty$ can be taken after the limit $N\to\infty$, this proves that the evolution of the distribution $D_t(\varsigma,h)$ is indeed deterministic on finite time-scales.

We now introduce the following set of order parameters (with their conjugates) in order to a achieve a factorization over sites of (18), by inserting appropriate integral representations of unity (from which all factors 2π will vanish in the limit $n\to 0$):

$$m_\alpha(\{\boldsymbol{\sigma}\}) = \frac{1}{N}\sum_i \sigma_i^\alpha \qquad W_{\alpha\beta}(\{\hat{\boldsymbol{h}},\boldsymbol{\sigma}\}) = \frac{1}{N}\sum_i \hat{h}_i^\alpha\hat{h}_i^\beta\sigma_i^\alpha\sigma_i^\beta$$

$$q_{\alpha\beta}(\{\boldsymbol{\sigma}\}) = \frac{1}{N}\sum_i \sigma_i^\alpha\sigma_i^\beta \qquad R_{\alpha\beta}(\{\hat{\boldsymbol{h}},\boldsymbol{\sigma}\}) = \frac{1}{N}\sum_i \hat{h}_i^\alpha\sigma_i^\beta \qquad Q_{\alpha\beta}(\{\hat{\boldsymbol{h}}\}) = \frac{1}{N}\sum_i \hat{h}_i^\alpha\hat{h}_i^\beta.$$

The δ-distribution involving $D_t(\varsigma,h)$ is also written in integral form. Combination of the trio (18), (19), (20) then leads to a fully site-factorized expression:

$$\begin{aligned}
\mathcal{A}[\varsigma,h;D] &= \mathrm{i}J^2\lim_{N\to\infty}\lim_{n\to 0}\int \mathrm{d}\boldsymbol{m}\,\mathrm{d}\hat{\boldsymbol{m}}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\hat{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\hat{\boldsymbol{Q}}\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\hat{\boldsymbol{R}}\,\mathrm{d}\boldsymbol{W}\,\mathrm{d}\hat{\boldsymbol{W}}\,\mathrm{d}\hat{\boldsymbol{D}} \\
&\quad\times\exp\left[J^2\sum_{\alpha\beta}W_{\alpha\beta}-\frac{N}{2}J^2\sum_{\alpha\beta}[Q_{\alpha\beta}q_{\alpha\beta}+R_{\alpha\beta}R_{\beta\alpha}]\right] \\
&\quad\times\exp\left\{\mathrm{i}N\sum_\alpha\left[\sum_{\varsigma'\mu}D(\varsigma',h_\mu)\hat{D}_\alpha(\varsigma',h_\mu)+m_\alpha\hat{m}_\alpha\right]\right. \\
&\quad\left.+\mathrm{i}N\sum_{\alpha\beta}[q_{\alpha\beta}\hat{q}_{\alpha\beta}+Q_{\alpha\beta}\hat{Q}_{\alpha\beta}+R_{\alpha\beta}\hat{R}_{\alpha\beta}+W_{\alpha\beta}\hat{W}_{\alpha\beta}]\right\} \\
&\quad\times\int[\mathrm{d}\boldsymbol{H}^1\,\mathrm{d}\hat{\boldsymbol{h}}^1]\cdots[\mathrm{d}\boldsymbol{H}^n\,\mathrm{d}\hat{\boldsymbol{h}}^n] \\
&\quad\times\sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n}\tanh(\beta H_1^1)\delta[h-H_2^1]\delta_{\varsigma,\sigma_2^1}\sum_\alpha[\hat{h}_2^\alpha\sigma_1^\alpha+\hat{h}_1^\alpha\sigma_2^\alpha] \\
&\quad\times\exp\left\{-\mathrm{i}\sum_i\sum_{\alpha\beta}[\hat{q}_{\alpha\beta}\sigma_i^\alpha\sigma_i^\beta+\hat{Q}_{\alpha\beta}\hat{h}_i^\alpha\hat{h}_i^\beta+\hat{R}_{\alpha\beta}\hat{h}_i^\alpha\sigma_i^\beta+\hat{W}_{\alpha\beta}\hat{h}_i^\alpha\hat{h}_i^\beta\sigma_i^\alpha\sigma_i^\beta]\right\} \\
&\quad\times\exp\left\{-\mathrm{i}\sum_i\sum_\alpha\left[\sum_\mu\hat{D}_\alpha(\sigma_i^\alpha,h_\mu)\delta[h_\mu-H_i^\alpha]+\hat{m}_\alpha\sigma_i^\alpha\right.\right. \\
&\quad\left.\left.-\hat{h}_i^\alpha\left[H_i^\alpha-\theta-J_0m_\alpha+\frac{J_0}{N}\sigma_i^\alpha\right]\right]\right\} \\
&= \mathrm{i}J^2\lim_{N\to\infty}\lim_{n\to 0}\int \mathrm{d}\boldsymbol{m}\,\mathrm{d}\hat{\boldsymbol{m}}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\hat{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\hat{\boldsymbol{Q}}\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\hat{\boldsymbol{R}}\,\mathrm{d}\boldsymbol{W}\,\mathrm{d}\hat{\boldsymbol{W}}\,\mathrm{d}\hat{\boldsymbol{D}}\,\mathrm{e}^{N\Psi+\mathcal{O}(1)} \\
&\quad\times\sum_\alpha\{\langle\tanh(\beta H_1)\sigma_\alpha\rangle_M\langle\delta[h-H_1]\delta_{\varsigma,\sigma_1}\hat{h}_\alpha\rangle_M \\
&\quad+\langle\tanh(\beta H_1)\hat{h}_\alpha\rangle_M\langle\delta[h-H_1]\delta_{\varsigma,\sigma_1}\sigma_\alpha\rangle_M\}
\end{aligned} \tag{22}$$

with

$$\Psi = \mathrm{i}\sum_{\alpha\sigma\mu} D(\sigma, h_\mu)\hat{D}_\alpha(\sigma, h_\mu) + \mathrm{i}\sum_\alpha m_\alpha \hat{m}_\alpha$$
$$+\mathrm{i}\sum_{\alpha\beta}[q_{\alpha\beta}\hat{q}_{\alpha\beta} + Q_{\alpha\beta}\hat{Q}_{\alpha\beta} + R_{\alpha\beta}\hat{R}_{\alpha\beta} + W_{\alpha\beta}\hat{W}_{\alpha\beta}]$$
$$-\frac{1}{2}J^2\sum_{\alpha\beta}[Q_{\alpha\beta}q_{\alpha\beta} + R_{\alpha\beta}R_{\beta\alpha}] + \log\int \mathrm{d}\boldsymbol{H}\,\mathrm{d}\hat{\boldsymbol{h}}\langle M[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}$$

(in which $\langle f(\boldsymbol{\sigma})\rangle_{\boldsymbol{\sigma}} = 2^{-n}\sum_{\sigma_1}\cdots\sum_{\sigma_n} f(\boldsymbol{\sigma})$) and with the effective single-site measure M (all vectors now carry replica-indices only):

$$\langle f[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}]\rangle_M = \frac{\int \mathrm{d}\boldsymbol{H}\,\mathrm{d}\hat{\boldsymbol{h}}\sum_{\boldsymbol{\sigma}} M[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}]f[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}]}{\int \mathrm{d}\boldsymbol{H}\,\mathrm{d}\hat{\boldsymbol{h}}\sum_{\boldsymbol{\sigma}} M[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}]} \tag{23}$$

$$M[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}] = \exp\Bigg\{-\mathrm{i}\hat{\boldsymbol{m}}\cdot\boldsymbol{\sigma} - \mathrm{i}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}}\boldsymbol{\sigma} - \mathrm{i}\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{Q}}\hat{\boldsymbol{h}} - \mathrm{i}\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{R}}\boldsymbol{\sigma} - \mathrm{i}\sum_{\alpha\beta}\hat{W}_{\alpha\beta}\hat{h}_\alpha\hat{h}_\beta\sigma_\alpha\sigma_\beta$$
$$-\sum_{\alpha\mu}\hat{D}_\alpha(\sigma_\alpha, h_\mu)\delta[h_\mu - H_\alpha] + \mathrm{i}\sum_\alpha \hat{h}_\alpha[H_\alpha - \theta - J_0 m_\alpha]\Bigg\}.$$

By changing the order of the limits $N \to \infty$ and $n \to 0$, the remaining integral can be evaluated by steepest descent. It is dominated by the extremum of Ψ which for $n > 1$ defines a global maximum (the $\mathcal{O}(1)$ term in the exponent in (22) will drop out due to normalization, as can be checked explicitly by using the above calculation to rewrite $\langle\langle 1\rangle\rangle_D$).

3.2. Simplification of the saddle-point problem

We can make several immediate simplifications. First, variation of Ψ with respect to $W_{\alpha\beta}$, $Q_{\alpha\beta}$, $q_{\alpha\beta}$ and $R_{\alpha\beta}$ gives saddle-point equations with which to remove all conjugate order parameter matrices from our problem:

$$\hat{\boldsymbol{W}} = 0 \qquad \mathrm{i}\hat{\boldsymbol{Q}} = \tfrac{1}{2}J^2\boldsymbol{q} \qquad \mathrm{i}\hat{\boldsymbol{q}} = \tfrac{1}{2}J^2\boldsymbol{Q} \qquad \mathrm{i}\hat{\boldsymbol{R}} = J^2\boldsymbol{R}^\dagger.$$

Second, the scaling freedom in the definition of the conjugate parameters $\hat{D}(\varsigma', h_\mu)$ can be used to take the limit $\ell \to \infty$:

$$\sum_\mu \hat{D}_\alpha(\sigma, h_\mu)f(h_\mu) \to \sum_\mu \Delta h\cdot\hat{D}_\alpha(\sigma, h_\mu)f(h_\mu) \to \int \mathrm{d}H\,\hat{D}_\alpha(\sigma, H)f(H)$$
$$(\ell \to \infty).$$

The result of these simplications and of taking the $N \to \infty$ limit is the following:

$$\mathcal{A}[\varsigma, h; D] = \mathrm{i}J^2\lim_{n\to 0}\sum_\alpha\{\langle\tanh(\beta H_1)\sigma_\alpha\rangle_M\langle\delta[h - H_1]\delta_{\varsigma,\sigma_1}\hat{h}_\alpha\rangle_M$$
$$+\langle\tanh(\beta H_1)\hat{h}_\alpha\rangle_M\langle\delta[h - H_1]\delta_{\varsigma,\sigma_1}\sigma_\alpha\rangle_M\}$$

with the effective measure (23), in which M and the exponent Ψ to be extremized are now given by

$$M[\boldsymbol{H}, \hat{\boldsymbol{h}}, \boldsymbol{\sigma}] = \exp\{-\mathrm{i}\hat{\boldsymbol{m}}\cdot\boldsymbol{\sigma} - \tfrac{1}{2}J^2\boldsymbol{\sigma}\cdot\boldsymbol{Q}\boldsymbol{\sigma} - \tfrac{1}{2}J^2\hat{\boldsymbol{h}}\cdot\boldsymbol{q}\hat{\boldsymbol{h}}$$
$$-\mathrm{i}\sum_\alpha\hat{D}_\alpha(\sigma_\alpha, H_\alpha) + \mathrm{i}\hat{\boldsymbol{h}}\cdot[\boldsymbol{H} - \boldsymbol{\theta} - J_0\boldsymbol{m} + \mathrm{i}J^2\boldsymbol{R}^\dagger\boldsymbol{\sigma}]\}$$

$$\Psi = \mathrm{i}\sum_{\alpha\sigma}\int \mathrm{d}H\,D(\sigma, H)\hat{D}_\alpha(\sigma, H) + \mathrm{i}\sum_\alpha m_\alpha\hat{m}_\alpha$$

$$+\frac{1}{2}J^2\sum_{\alpha\beta}[q_{\alpha\beta}Q_{\alpha\beta}+R_{\alpha\beta}R_{\beta\alpha}]+\log\int \mathrm{d}\boldsymbol{H}\,\mathrm{d}\hat{\boldsymbol{h}}\langle M[\boldsymbol{H},\hat{\boldsymbol{h}},\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}$$

with the notation $\boldsymbol{\theta}=(\theta,\ldots,\theta)$. Next we perform the integrations over the conjugate fields $\hat{\boldsymbol{h}}$, which leads to an effective measure M involving spins and fields only:

$$\langle f[\boldsymbol{H},\boldsymbol{\sigma}]\rangle_M=\frac{\int \mathrm{d}\boldsymbol{H}\sum_{\boldsymbol{\sigma}}M[\boldsymbol{H},\boldsymbol{\sigma}]f[\boldsymbol{H},\boldsymbol{\sigma}]}{\int \mathrm{d}\boldsymbol{H}\sum_{\boldsymbol{\sigma}}M[\boldsymbol{H},\boldsymbol{\sigma}]} \tag{24}$$

$$M[\boldsymbol{H},\boldsymbol{\sigma}]=\exp\Bigg\{-\mathrm{i}\hat{\boldsymbol{m}}\cdot\boldsymbol{\sigma}-\frac{1}{2}J^2\boldsymbol{\sigma}\cdot\boldsymbol{Q}\boldsymbol{\sigma}-\mathrm{i}\sum_\alpha\hat{D}_\alpha(\sigma_\alpha,H_\alpha)$$
$$-\frac{1}{2J^2}[\boldsymbol{H}-\boldsymbol{\theta}-J_0\boldsymbol{m}+\mathrm{i}J^2\boldsymbol{R}^\dagger\boldsymbol{\sigma}]\cdot\boldsymbol{q}^{-1}[\boldsymbol{H}-\boldsymbol{\theta}-J_0\boldsymbol{m}+\mathrm{i}J^2\boldsymbol{R}^\dagger\boldsymbol{\sigma}]\Bigg\} \tag{25}$$

$$\Psi=\mathrm{i}\sum_{\alpha\sigma}\int \mathrm{d}H\,D(\sigma,H)\hat{D}_\alpha(\sigma,H)+\mathrm{i}\sum_\alpha m_\alpha\hat{m}_\alpha+\frac{1}{2}J^2\sum_{\alpha\beta}[q_{\alpha\beta}Q_{\alpha\beta}+R_{\alpha\beta}R_{\beta\alpha}]$$
$$-\frac{1}{2}\log\det\boldsymbol{q}+\log\int \mathrm{d}\boldsymbol{H}\,\langle M[\boldsymbol{H},\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}. \tag{26}$$

In Ψ (26) we have neglected irrelevant constants. At this stage it will be convenient to calculate the remaining saddle-point equations, by variation of (26). The first of these equations, obtained by variation with respect to $\hat{D}(\sigma,H)$, enables us to write all averages with a single replica-index, involving fields and spins, self-consistently in terms of the original distribution $D(\sigma,H)$:

$$D(\sigma,H)=\langle\delta_{\sigma,\sigma_\alpha}\delta[H-H_\alpha]\rangle_M \tag{27}$$

$$m_\alpha=m=\langle\sigma\rangle_D \tag{28}$$

$$q_{\alpha\beta}=\langle\sigma_\alpha\sigma_\beta\rangle_M \tag{29}$$

$$\hat{m}_\alpha=\mathrm{i}\frac{J_0}{J^2}\sum_\beta(q^{-1})_{\alpha\beta}\Bigg\{\langle H\rangle_D-\theta-J_0m+\mathrm{i}J^2m\sum_\gamma R_{\gamma\beta}\Bigg\} \tag{30}$$

$$R_{\alpha\beta}=\frac{\mathrm{i}}{J^2}\sum_\gamma(q^{-1})_{\alpha\gamma}\langle[H_\gamma-\theta-J_0m+\mathrm{i}J^2(\boldsymbol{R}^\dagger\boldsymbol{\sigma})_\gamma]\sigma_\beta\rangle_M \tag{31}$$

$$J^2Q_{\alpha\beta}=\frac{\partial}{\partial q_{\alpha\beta}}\log\det\boldsymbol{q}-2\left(\frac{\partial\log M[\boldsymbol{H},\boldsymbol{\sigma}]}{\partial q_{\alpha\beta}}\right)_M. \tag{32}$$

We can now write the flow term $\mathcal{A}$ (18) of our diffusion equation as

$$\mathcal{A}[\varsigma,h;D]=-\lim_{n\to0}\sum_{\alpha\beta}(q^{-1})_{\alpha\beta}$$
$$\times\{\langle\tanh(\beta H_1)\sigma_\alpha\rangle_M\langle\delta[h-H_1]\delta_{\varsigma,\sigma_1}[H_\beta-\theta-J_0m+\mathrm{i}J^2(\boldsymbol{R}^\dagger\boldsymbol{\sigma})_\beta]\rangle_M$$
$$+\langle\delta[h-H_1]\delta_{\varsigma,\sigma_1}\sigma_\alpha\rangle_M\langle\tanh(\beta H_1)[H_\beta-\theta-J_0m+\mathrm{i}J^2(\boldsymbol{R}^\dagger\boldsymbol{\sigma})_\beta]\rangle_M\}. \tag{33}$$

3.3. *Equilibrium*

In equilibrium, we know that the microscopic probability distribution is of the Boltzmann form, $p_\infty(\boldsymbol{\sigma})\sim \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}$. Therefore, the present constraint restricting micro-states under consideration to those with the same joint spin-field distribution, must in equilibrium reduce to a constraint selecting states with the same energy. We will now make the ansatz

$$\hat{D}_\alpha(\sigma,H)=\tfrac{1}{2}\mathrm{i}\beta\sigma[H+\theta] \tag{34}$$

and show that it indeed corresponds to a stationary state for our diffusion equation (17), in which one recovers the familiar equations from equilibrium statistical mechanics; i.e. the full (RSB) order parameter equations [2, 3] as well as the equilibrium local field distribution [8].

We first turn to the saddle-point equations. Given the simple expression (34) we can perform the field integrations, with the result

$$\Psi = -\frac{1}{2}\beta n \sum_\sigma \int \mathrm{d}H\, D(\sigma, H)\sigma[H+\theta] + \mathrm{i}\sum_\alpha m_\alpha \hat{m}_\alpha + \frac{1}{2}J^2 \sum_{\alpha\beta}[q_{\alpha\beta}Q_{\alpha\beta} + R_{\alpha\beta}R_{\beta\alpha}]$$
$$+ \log\langle \exp\{[\beta\boldsymbol{\theta} + \tfrac{1}{2}\beta J_0 \boldsymbol{m} - \mathrm{i}\hat{\boldsymbol{m}}]\cdot\boldsymbol{\sigma} + \tfrac{1}{2}J^2\boldsymbol{\sigma}\cdot[\tfrac{1}{4}\beta^2\boldsymbol{q} - \boldsymbol{Q} - \mathrm{i}\beta\boldsymbol{R}^\dagger]\boldsymbol{\sigma}\}\rangle_{\boldsymbol{\sigma}} \tag{35}$$

(again we forget about irrelevant constants). The remaining saddle-point equations become

$$\hat{m}_\alpha = \tfrac{1}{2}\mathrm{i}\beta J_0 m_\alpha \qquad Q_{\alpha\beta} = -\tfrac{1}{4}\beta^2 q_{\alpha\beta} \qquad R_{\alpha\beta} = \tfrac{1}{2}\mathrm{i}\beta q_{\beta\alpha} \tag{36}$$

$$m_\alpha = \frac{\langle \sigma_\alpha \exp[\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}}{\langle \exp[\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}}$$
$$q_{\alpha\beta} = \frac{\langle \sigma_\alpha\sigma_\beta \exp[\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}}{\langle \exp[\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}}} \tag{37}$$

which are the familiar equations [3] as obtained by an equilibrium (thermodynamic) analysis. With the relations (34), (36) we can simplify the effective measure M considerably:

$$\langle f[\boldsymbol{H},\boldsymbol{\sigma}]\rangle_M = \frac{\int \mathrm{d}\boldsymbol{z}\, \exp(-\frac{1}{2}\boldsymbol{z}\cdot\boldsymbol{q}^{-1}\boldsymbol{z})\langle f[J_0\boldsymbol{m}+\boldsymbol{\theta}+J\boldsymbol{z},\boldsymbol{\sigma}]\exp(\beta\boldsymbol{\sigma}\cdot[J_0\boldsymbol{m}+\boldsymbol{\theta}+J\boldsymbol{z}])\rangle_{\boldsymbol{\sigma}}}{\int \mathrm{d}\boldsymbol{z}\, \exp(-\frac{1}{2}\boldsymbol{z}\cdot\boldsymbol{q}^{-1}\boldsymbol{z})\langle \exp(\beta\boldsymbol{\sigma}\cdot[J_0\boldsymbol{m}+\boldsymbol{\theta}+J\boldsymbol{z}])\rangle_{\boldsymbol{\sigma}}} \tag{38}$$

(with $\boldsymbol{m} = (m, \ldots, m)$) This simplified measure obeys useful relations like

$$\langle \sigma_\alpha f[\boldsymbol{H}; \{\sigma_{\gamma\neq\alpha}\}]\rangle_M = \langle \tanh(\beta H_\alpha) f[\boldsymbol{H}; \{\sigma_{\gamma\neq\alpha}\}]\rangle_M \tag{39}$$

$$\langle (H_\alpha - J_0 m - \theta) f[\{H_{\gamma\neq\alpha}\}; \boldsymbol{\sigma}]\rangle_M = \beta J^2 \sum_\beta q_{\alpha\beta}\langle \sigma_\beta f[\{H_{\gamma\neq\alpha}\}; \boldsymbol{\sigma}]\rangle_M. \tag{40}$$

In particular we now find $m = \langle \tanh(\beta H)\rangle_D$. If we combine the expression (38) with (27), sum over the remaining spin variable σ and perform the integration over $\boldsymbol{z}$, we are led directly to the equilibrium expression for the local field distribution as obtained in [8]:

$$D(h) = \lim_{n\to 0}\int \frac{\mathrm{d}k}{2\pi}\exp\{-\tfrac{1}{2}J^2k^2 - \mathrm{i}k(h - J_0 m - \theta)\}$$
$$\times \frac{\langle \exp\{\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma} + \mathrm{i}k\beta J^2\sum_\alpha q_{1\alpha}\sigma_\alpha\}\rangle_{\boldsymbol{\sigma}}}{\langle \exp\{\beta[J_0\boldsymbol{m}+\boldsymbol{\theta}]\cdot\boldsymbol{\sigma} + \frac{1}{2}(\beta J)^2\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}\}\rangle_{\boldsymbol{\sigma}}}.$$

Next we show that the choice (34) corresponds to a fixed point of the diffusion equation (17), i.e. that $\frac{\mathrm{d}}{\mathrm{d}t}D_t(\varsigma, h) = 0$ for all (ς, h). In the right-hand side of (17) the first two terms trivially cancel, which follows from applying to (27) the identities $\delta_{\varsigma,\sigma} = \frac{1}{2}[1+\varsigma\sigma]$ and (39):

$$[1+\varsigma\tanh(\beta h)]D(-\varsigma, h) - [1-\varsigma\tanh(\beta h)]D(\varsigma, h)$$
$$= \varsigma\langle \delta[h - H_\alpha][\tanh(\beta h) - \sigma_\alpha]\rangle_M = 0.$$

Equivalently:

$$D(\varsigma, h) = \tfrac{1}{2}[1+\varsigma\tanh(\beta h)]D(h)$$

We use (36) and (39), (40) to rewrite (33). In doing so we will also use equilibrium relations like

$$\beta J^2 \sum_\alpha q_{1\alpha}^2 = \langle \tanh(\beta H)(H - J_0 m - \theta)\rangle_D$$

which can be derived directly from the equilibrium saddle-point equations (see e.g. [1]). The result is:

$$\mathcal{A}[\varsigma, h; D] = -(h - \theta - J_0 m) D(\varsigma, h) - \beta J^2 [1 - \langle \tanh^2(\beta H)\rangle_D] \varsigma D(\varsigma.h) \\ +[1 - \langle \tanh^2(\beta H)\rangle_D] \lim_{n\to 0} \sum_\gamma (q^{-1})_{1\gamma} \langle \delta[h - H_1] \delta_{\varsigma,\sigma_1} [H_\gamma - J_0 m - \theta]\rangle_M. \tag{41}$$

In order to combine the flow term $\mathcal{A}$ in (17) with the diffusion term, we apply (38) to equation (27) and calculate the field derivative:

$$J^2 \frac{\partial}{\partial h} D(\varsigma, h) = J \lim_{n\to 0} \left[\int \mathrm{d}\boldsymbol{z}\, \delta[h - J_0 m - \theta - J z_\alpha] \frac{\partial}{\partial z_\alpha} \right. \\ \left. \times \{\exp\{-\tfrac{1}{2} \boldsymbol{z} \cdot \boldsymbol{q}^{-1} \boldsymbol{z}\} \langle \delta_{\varsigma,\sigma_\alpha} \exp\{\beta \boldsymbol{\sigma}[J_0 \boldsymbol{m} + \boldsymbol{\theta} + J \boldsymbol{z}]\}\rangle_{\boldsymbol{\sigma}}\} \right] \\ \times \left[\int \mathrm{d}\boldsymbol{z}\, \exp\{-\tfrac{1}{2} \boldsymbol{z} \cdot \boldsymbol{q}^{-1} \boldsymbol{z}\} \langle \exp\{\beta \boldsymbol{\sigma}[J_0 \boldsymbol{m} + \boldsymbol{\theta} + J \boldsymbol{z}]\}\rangle_{\boldsymbol{\sigma}} \right]^{-1} \\ = \beta J^2 \varsigma D(\varsigma, h) - \lim_{n\to 0} \sum_\gamma (q^{-1})_{1\gamma} \langle \delta[h - H_1] \delta_{\varsigma,\sigma_1} [H_\gamma - J_0 m - \theta]\rangle_M. \tag{42}$$

Insertion of (41) and (42) into the right-hand side of (17) leads to the desired result: it exactly vanishes. This completes the proof that the standard thermodynamic equilibrium state, as calculated within equilibrium statistical mechanics, defines a fixed-point of our diffusion equation (17). Note, however, that this leaves open the possibility of existence for stationary states other than the thermodynamic one.

4. Replica symmetric flow

4.1. Derivation of the RS equations

In order to proceed further in evaluating explicitly the saddle points we now make, as a first step, the ergodicity or replica-symmetry ansatz (RS). All order parameters with a single replica index are assumed not to depend on this index; all order parameter matrices are assumed to have entries which depend only on whether or not they are on the diagonal. With a modest amount of foresight we put

$$m_\alpha = m \qquad q_{\alpha\beta} = (1-q)\delta_{\alpha\beta} + q \\ \hat{m}_\alpha = i\mu \qquad R_{\alpha\beta} = \mathrm{i}(1-q)[R_0 \delta_{\alpha\beta} + R] \\ \hat{D}_\alpha(\sigma, H) = \mathrm{i} \log \chi(\sigma, H) \qquad Q_{\alpha\beta} = Q_0 \delta_{\alpha\beta} + q R_0^2 - 2(1-q) R R_0 - Q^2 \tag{43}$$

which implies $(q^{-1})_{\alpha\beta} = (1-q)^{-1}[\delta_{\alpha\beta} - q(1-q)^{-1}] + \mathcal{O}(n)$. Working out the RS version of the extensive exponent Ψ (26) gives

$$\lim_{n\to 0} \frac{\Psi_{\mathrm{RS}}}{n} = -m\mu - \frac{1}{2}\log(1-q) - \frac{q}{2(1-q)} - \frac{1}{2} J^2 (1-q) Q^2 - J^2 (1-q)^2 [R_0^2 + 2 R_0 R] \\ - \sum_\sigma \int \mathrm{d}H\, D(\sigma, H) \log \chi(\sigma, H) + \lim_{n\to 0} \frac{1}{n} \log \int \mathrm{d}\boldsymbol{H}\, \langle M_{\mathrm{RS}}[\boldsymbol{H}, \boldsymbol{\sigma}]\rangle_{\boldsymbol{\sigma}} \tag{44}$$

with

$$M_{\rm RS}[\boldsymbol{H},\boldsymbol{\sigma}] = \prod_\alpha \Bigg\{\chi(\sigma_\alpha, H_\alpha) \times \exp\left[\mu\sigma_\alpha - \frac{1}{2J^2(1-q)}(H_\alpha-\theta-J_0m)^2 + R_0(H_\alpha-\theta-J_0m)\sigma_\alpha\right]\Bigg\} \times \exp\Bigg\{\frac{q}{2J^2(1-q)^2}\left[\sum_\alpha(H_\alpha-\theta-J_0m)\right]^2 + \frac{1}{2}J^2Q^2\left[\sum_\alpha\sigma_\alpha\right]^2 + \left(R-\frac{qR_0}{1-q}\right)\left[\sum_\alpha(H_\alpha-\theta-J_0m)\right]\left[\sum_\beta\sigma_\beta\right]\Bigg\}. \tag{45}$$

We can obtain a factorization of $M_{\rm RS}[\boldsymbol{H},\boldsymbol{\sigma}]$ with respect to the replica labels by introducing appropriate Gaussian integrations:

$$\exp\left\{A\left[\sum_\alpha F_\alpha\right]^2 + B\left[\sum_\alpha\sigma_\alpha\right]^2 + C\left[\sum_\alpha F_\alpha\right]\left[\sum_\beta\sigma_\beta\right]\right\} = \int {\rm D}x\,{\rm D}y\prod_\alpha \exp\{F_\alpha\sqrt{2A}(x\cos\phi+y\sin\phi)+\sigma_\alpha\sqrt{2B}(x\cos\phi-y\sin\phi)\}$$

with $\cos(2\phi)=C/2\sqrt{AB}$ and $Dx=(2\pi)^{-1/2}\,{\rm e}^{-x^2/2}\,{\rm d}x$. Application of the above identity to (45) leads to an expression for (44) in which we can take the remaining limit $n\to 0$. We use the definition of the angle ϕ to eliminate the order parameter R from our problem and write the averages over the two Gaussian variables x and y as $\langle\!\langle\ldots\rangle\!\rangle_{xy}$. The final result involves an effective measure $M_{\rm RS}[H,\sigma]$ without replica indices:

$$\lim_{n\to 0}\frac{\Psi_{\rm RS}}{n} = -m\mu - \frac{1}{2}\log(1-q) - \frac{q}{2(1-q)} - J^2(1-q)\left[\frac{1}{2}Q^2+(1+q)R_0^2+2R_0Q\sqrt{q}\cos(2\phi)\right] - \sum_\sigma\int{\rm d}H\,D(\sigma,H)\log\chi(\sigma,H) + \left\langle\!\!\left\langle\log\int{\rm d}H\,\langle M_{\rm RS}[H,\sigma]\rangle_\sigma\right\rangle\!\!\right\rangle_{xy} \tag{46}$$

with

$$M_{\rm RS}[H,\sigma] = \chi(\sigma,H)\exp\left\{\mu\sigma - \frac{(H-\theta-J_0m)^2}{2J^2(1-q)} + R_0(H-\theta-J_0m)\sigma + \frac{\sqrt{q}}{J(1-q)}(H-\theta-J_0m)(x\cos\phi+y\sin\phi) + JQ\sigma(x\cos\phi-y\sin\phi)\right\}. \tag{47}$$

We write averages with respect to this final measure $M_{\rm RS}[H,\sigma]$, which are parametrized by the Gaussian variables x and y, as

$$\langle f[H,\sigma]\rangle_\star = \frac{\int{\rm d}H\sum_\sigma M_{\rm RS}[H,\sigma]f[H,\sigma]}{\int{\rm d}H\sum_\sigma M_{\rm RS}[H,\sigma]}.$$

To further reduce our future bookkeeping we derive two useful relations by partial integration over the Gaussian variables:

$$\langle\!\langle x\langle f[H,\sigma]\rangle_\star\rangle\!\rangle_{xy} = \frac{\cos\phi\sqrt{q}}{J(1-q)}\langle\!\langle\langle f[H,\sigma]H\rangle_\star - \langle f[H,\sigma]\rangle_\star\langle H\rangle_\star\rangle\!\rangle_{xy} + JQ\cos\phi\langle\!\langle\langle f[H,\sigma]\sigma\rangle_\star - \langle f[H,\sigma]\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy}$$

$$\langle\!\langle y\langle f[H,\sigma]\rangle_\star\rangle\!\rangle_{xy} = \frac{\sin\phi\sqrt{q}}{J(1-q)}\langle\!\langle\langle f[H,\sigma]H\rangle_\star - \langle f[H,\sigma]\rangle_\star\langle H\rangle_\star\rangle\!\rangle_{xy}$$
$$-JQ\sin\phi\langle\!\langle\langle f[H,\sigma]\sigma\rangle_\star - \langle f[H,\sigma]\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy}.$$

Functional differentiation of (46) with respect to the function χ gives the RS saddle-point equation

$$D(\varsigma,h) = \langle\!\langle\langle\delta[h-H]\delta_{\varsigma,\sigma}\rangle_\star\rangle\!\rangle_{xy} \tag{48}$$

which implies, as expected, the general relation

$$\langle\!\langle\langle f[H,\sigma]\rangle_\star\rangle\!\rangle_{xy} = \langle f[H,\sigma]\rangle_D.$$

Differentiation of (46) with respect to the parameters $\{q, m, \mu, R_0, Q, \phi\}$ and repeated usage of the above bookkeeping identities gives the remaining RS saddle-point equations:

$$m = \langle\sigma\rangle_D \tag{49}$$
$$q = \langle\!\langle\langle\sigma\rangle_\star^2\rangle\!\rangle_{xy} \tag{50}$$
$$2J^2R_0(1-q)^2 = \langle\sigma(H-J_0m-\theta)\rangle_D - \langle\!\langle\langle\sigma\rangle_\star\langle H-J_0m-\theta\rangle_\star\rangle\!\rangle_{xy} \tag{51}$$
$$2J^2(1-q)[R_0(1+q)+Q\sqrt{q}\cos(2\phi)] = \langle\sigma(H-J_0m-\theta)\rangle_D \tag{52}$$
$$\mu + J_0R_0m = \frac{J_0}{J^2(1-q)}\langle H-J_0m-\theta\rangle_D \tag{53}$$
$$\begin{aligned}J^2q - J^4(1-q)^2[Q^2+4qR_0^2+8QR_0\sqrt{q}\cos(2\phi)] + \langle(H-J_0m-\theta)^2\rangle_D \\ = \frac{1+q}{1-q}\langle\!\langle\langle(H-J_0m-\theta)^2\rangle_\star - \langle H-J_0m-\theta\rangle_\star^2\rangle\!\rangle_{xy}.\end{aligned} \tag{54}$$

We now use the RS ansatz to perform the $n \to 0$ limit in the flow term $\mathcal{A}$ (33) of our diffusion equation (17). Note that, due to $n \to 0$, we may deal with averages over the original measure M which involve two replica indices (such as those encountered in (33) in the following way:

$$\langle f[H_\alpha,\sigma_\alpha]g[H_\beta,\sigma_\beta]\rangle_M \to \delta_{\alpha\beta}\langle f[H,\sigma]g[H,\sigma]\rangle_D + (1-\delta_{\alpha\beta})\langle\!\langle\langle f[H,\sigma]\rangle_\star\langle g[H,\sigma]\rangle_\star\rangle\!\rangle_{xy}.$$

With this identity we can work out (33). We use the short-hand $Q\sqrt{q}\cos(2\phi) = (1-q)R_1$, and find after some bookkeeping and some re-arranging of terms:

$$\begin{aligned}(1-q)^2\mathcal{A}_{\rm RS}[\varsigma,h;D] = {} & (2q-1)D(\varsigma,h) \\ & \times[(h-J_0m-\theta)\langle\tanh(\beta H)\sigma\rangle_D + \varsigma\langle\tanh(\beta H)(H-J_0m-\theta)\rangle_D] \\ & -qD(\varsigma,h)[(h-J_0m-\theta)\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy} \\ & +\varsigma\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle H-J_0m-\theta\rangle_\star\rangle\!\rangle_{xy}] \\ & +2\varsigma J^2(1-q)^2D(\varsigma,h) \\ & \times[(R_1+R_0)\langle\tanh(\beta H)\sigma\rangle_D - R_1\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy}] \\ & +\langle\!\langle\langle\delta[h-H]\delta_{\varsigma,\sigma}\rangle_\star\langle H-J_0m-\theta\rangle_\star\rangle\!\rangle_{xy} \\ & \times[\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy} - q\langle\tanh(\beta H)\sigma\rangle_D] \\ & +\langle\!\langle\langle\delta[h-H]\delta_{\varsigma,\sigma}\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy} \\ & \times[\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle H-J_0m-\theta\rangle_\star\rangle\!\rangle_{xy} - q\langle\tanh(\beta H)(H-J_0m-\theta)\rangle_D] \\ & +2J^2(1-q)^2\langle\!\langle\langle\delta[h-H]\delta_{\varsigma,\sigma}\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy} \\ & \times[(R_1-R_0)\langle\!\langle\langle\tanh(\beta H)\rangle_\star\langle\sigma\rangle_\star\rangle\!\rangle_{xy} - R_1\langle\tanh(\beta H)\sigma\rangle_D].\end{aligned} \tag{55}$$

In the RS approximation the evolution of the joint spin-field distribution is described by equation (17), in which the disorder-induced term $\mathcal{A}$ is given by (55). Evaluation of $\mathcal{A}$, in turn, requires solving the set of saddle-point equations (48)–(54), at each instance of time.

4.2. The AT instability

The de Almeida–Thouless (AT) instability [9] marks the instability of the RS solution of the saddle-point equations to the so-called replicon mode. This leads to a second-order transition away from the RS state to states with broken replica symmetry (RSB). Unlike the standard equilibrium calculations, we here have to worry about replicon fluctuations with respect to three replica matrices:

$$\begin{aligned} &q_{\alpha\beta} \to q_{\alpha\beta}^{\mathrm{RS}} + \delta q_{\alpha\beta} \qquad \delta q_{\alpha\alpha} = 0 \qquad \sum_\alpha \delta q_{\alpha\beta} = \sum_\beta \delta q_{\alpha\beta} = 0 \\ &Q_{\alpha\beta} \to Q_{\alpha\beta}^{\mathrm{RS}} + \delta Q_{\alpha\beta} \qquad \delta Q_{\alpha\alpha} = 0 \qquad \sum_\alpha \delta Q_{\alpha\beta} = \sum_\beta \delta Q_{\alpha\beta} = 0 \\ &R_{\alpha\beta} \to R_{\alpha\beta}^{\mathrm{RS}} + \mathrm{i}\delta R_{\alpha\beta} \qquad \delta R_{\alpha\alpha} = 0 \qquad \sum_\alpha \delta R_{\alpha\beta} = \sum_\beta \delta R_{\alpha\beta} = 0 \end{aligned} \tag{56}$$

with $\delta q_{\alpha\beta} = \delta q_{\beta\alpha}$, $\delta Q_{\alpha\beta} = \delta Q_{\beta\alpha}$ and $\delta R_{\alpha\beta} = \delta R_{\beta\alpha}$. As usual the replicon fluctuations satisfy convenient matrix commutation relations, like $[\boldsymbol{q}_{\mathrm{RS}}, \delta\boldsymbol{q}] = [\boldsymbol{Q}_{\mathrm{RS}}, \delta\boldsymbol{Q}] = [\boldsymbol{R}_{\mathrm{RS}}, \delta\boldsymbol{R}] = 0$. The AT instability corresponds to a zero eigenvalue in the spectrum of the Hessian (i.e. the matrix of second derivatives) of Ψ at the RS saddle-point. However, since the $R_{\alpha\beta}$ are conjugate order parameters, acquiring an imaginary value, the naive picture of this zero eigenvalue signalling the bifurcation of a local maximum, need not be true. We can avoid all such subtleties by following the alternative procedure: to consider RSB fluctuations only after elimination of the conjugate order parameters $R_{\alpha\beta}$ with equation (31). This is equivalent to first working out the variation in Ψ (26) for the case where all fluctuations (56) are independent, followed by a projection onto the subspace defined by (31).

Expansion of (26) around the RS saddle point, the first non-trivial order of which must by definition be quadratic in the replicon fluctuations, gives

$$\begin{aligned} \Psi - \Psi_{\mathrm{RS}} = {}& \frac{1}{2}\langle G^2\rangle_M + \frac{1}{2}J^2 \sum_{\alpha\beta} \delta q_{\alpha\beta}\delta Q_{\alpha\beta} - J^2 \sum_{\alpha\beta} \delta R_{\alpha\beta}^2 - J^2 R_0 \sum_{\alpha\beta} \delta R_{\alpha\beta}\delta q_{\alpha\beta} \\ & + \frac{1}{2}\sum_{\alpha\beta} \delta q_{\alpha\beta}^2 \left[\frac{1}{2(1-q)^2} + 3J^2R_0^2 - \frac{1}{J^2(1-q)^3} \langle\!\langle \langle H^2\rangle_\star - \langle H\rangle_\star^2 \rangle\!\rangle_{xy} \right] \\ & + \mathcal{O}(\delta^3) \end{aligned} \tag{57}$$

with

$$\begin{aligned} G = {}& -\frac{1}{2}J^2 \boldsymbol{\sigma}\cdot[\delta\boldsymbol{Q} + 2R_0\delta\boldsymbol{R} - R_0^2\delta\boldsymbol{q}]\boldsymbol{\sigma} + \frac{(\boldsymbol{H} - J_0\boldsymbol{m} - \boldsymbol{\theta})\cdot\delta\boldsymbol{q}(\boldsymbol{H} - J_0\boldsymbol{m} - \boldsymbol{\theta})}{2J^2(1-q)^2} \\ & + \frac{\boldsymbol{\sigma}\cdot[\delta\boldsymbol{R} - R_0\delta\boldsymbol{q}](\boldsymbol{H} - J_0\boldsymbol{m} - \boldsymbol{\theta})}{1-q}. \end{aligned}$$

In order to evaluate the term in (57) that involves G, we note that in RS saddle-points and for indices $\alpha \neq \beta$ and $\gamma \neq \lambda$:

$$\begin{aligned} \langle f_\alpha g_\beta h_\gamma k_\lambda\rangle_M = {}& \delta_{\alpha\gamma}\delta_{\beta\lambda} \langle\!\langle \langle fh\rangle_\star \langle gk\rangle_\star + \langle f\rangle_\star\langle g\rangle_\star\langle h\rangle_\star\langle k\rangle_\star \\ & - \langle fh\rangle_\star\langle g\rangle_\star\langle k\rangle_\star - \langle gk\rangle_\star\langle f\rangle_\star\langle h\rangle_\star \rangle\!\rangle_{xy} \\ & + \delta_{\alpha\lambda}\delta_{\beta\gamma} \langle\!\langle \langle fk\rangle_\star\langle gh\rangle_\star + \langle f\rangle_\star\langle g\rangle_\star\langle h\rangle_\star\langle k\rangle_\star \\ & - \langle fk\rangle_\star\langle g\rangle_\star\langle h\rangle_\star - \langle gh\rangle_\star\langle f\rangle_\star\langle k\rangle_\star \rangle\!\rangle_{xy} \\ & + \text{ terms with less than two } \delta\text{'s}. \end{aligned}$$

Only the terms with two Kronecker δ's can contribute to $\langle G^2\rangle_M$, due to the specific properties of the replicon fluctuations. We now obtain

$$
\begin{aligned}
\frac{1}{2}\langle G^2\rangle_M = \frac{1}{4}J^4 \langle\!\langle [1-\langle\sigma\rangle_\star^2]^2\rangle\!\rangle_{xy} \sum_{\alpha\beta}[\delta Q_{\alpha\beta}+2R_0\delta R_{\alpha\beta}-R_0^2\delta q_{\alpha\beta}]^2 \\
&+\frac{1}{4J^4(1-q)^4}\langle\!\langle[\langle H^2\rangle_\star-\langle H\rangle_\star^2]^2\rangle\!\rangle_{xy} \\
&\times\sum_{\alpha\beta}\delta q_{\alpha\beta}^2 \\
&+\frac{1}{2(1-q)^2}\langle\!\langle[1-\langle\sigma\rangle_\star^2][\langle H^2\rangle_\star-\langle H\rangle_\star^2] \\
&+[\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star]^2\rangle\!\rangle_{xy} \\
&\times\sum_{\alpha\beta}[\delta R_{\alpha\beta}-R_0\delta q_{\alpha\beta}]^2 \\
&-\frac{1}{2(1-q)^2}\langle\!\langle[\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star\rangle]^2\rangle\!\rangle_{xy} \\
&\times\sum_{\alpha\beta}\delta q_{\alpha\beta}[\delta Q_{\alpha\beta}+2R_0\delta R_{\alpha\beta}-R_0^2\delta q_{\alpha\beta}] \\
&-\frac{J^2}{1-q}\langle\!\langle[1-\langle\sigma\rangle_\star^2][\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star]\rangle\!\rangle_{xy} \\
&\times\sum_{\alpha\beta}[\delta R_{\alpha\beta}-R_0\delta q_{\alpha\beta}][\delta Q_{\alpha\beta}+2R_0\delta R_{\alpha\beta}-R_0^2\delta q_{\alpha\beta}] \\
&+\frac{1}{J^2(1-q)^3}\langle\!\langle[\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star][\langle H^2\rangle_\star-\langle H\rangle_\star^2]\rangle\!\rangle_{xy} \\
&\times\sum_{\alpha\beta}\delta q_{\alpha\beta}[\delta R_{\alpha\beta}-R_0\delta q_{\alpha\beta}].
\end{aligned}
$$

The various combinations of matrix fluctuations can be somewhat disentangled by introducing the transformation

$$\delta\boldsymbol{Q}=-R_0^2\delta\boldsymbol{q}-2\frac{R_0}{J}\delta\boldsymbol{r}+\frac{2}{J^2}\delta\boldsymbol{k} \qquad \delta\boldsymbol{R}=R_0\delta\boldsymbol{q}+\frac{1}{J}\delta\boldsymbol{r}.$$

In addition this renders all fluctuations dimensionless. Expression (57) now acquires the form

$$\Psi-\Psi_{\mathrm{RS}}=\sum_{\alpha\beta}\begin{pmatrix}\delta k_{\alpha\beta}\\ \delta q_{\alpha\beta}\\ \delta r_{\alpha\beta}\end{pmatrix}\boldsymbol{M}\begin{pmatrix}\delta k_{\alpha\beta}\\ \delta q_{\alpha\beta}\\ \delta r_{\alpha\beta}\end{pmatrix}+\mathcal{O}(\delta^3) \tag{58}$$

in which the entries of the symmetric 3×3 matrix $\boldsymbol{M}$ are

$$M_{11}=\langle\!\langle[1-\langle\sigma\rangle_\star^2]^2\rangle\!\rangle_{xy}$$

$$M_{12}=M_{21}=\frac{1}{2}-\frac{1}{2J^2(1-q)^2}\langle\!\langle[\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star]^2\rangle\!\rangle_{xy}$$

$$M_{13}=M_{31}=-\frac{1}{J(1-q)}\langle\!\langle[1-\langle\sigma\rangle_\star^2][\langle\sigma H\rangle_\star-\langle\sigma\rangle_\star\langle H\rangle_\star]\rangle\!\rangle_{xy}$$

$$M_{22}=\frac{1}{4(1-q)^2}\langle\!\langle[1-\frac{\langle H^2\rangle_\star-\langle H\rangle_\star^2}{J^2(1-q)}]^2\rangle\!\rangle_{xy}-J^2R_0^2$$

$$M_{23} = M_{32} = \frac{1}{2J^3(1-q)^3}\langle\!\langle[\langle\sigma H\rangle_\star - \langle\sigma\rangle_\star\langle H\rangle_\star][\langle H^2\rangle_\star - \langle H\rangle_\star^2]\rangle\!\rangle_{xy} - 2JR_0$$

$$M_{33} = \frac{1}{2J^2(1-q)^2}\langle\!\langle[1-\langle\sigma\rangle_\star^2][\langle H^2\rangle_\star - \langle H\rangle_\star^2] + [\langle\sigma H\rangle_\star - \langle\sigma\rangle_\star\langle H\rangle_\star]^2\rangle\!\rangle_{xy} - 1. \tag{59}$$

We now use (31) to eliminate the conjugate order parameters $R_{\alpha\beta}$ from our equations. In the space of RS saddle-points and replicon fluctuations we satisfy $[\boldsymbol{R}, \boldsymbol{q}] = 0$, so equation (31) simplifies to

$$(\boldsymbol{q}\boldsymbol{R})_{\alpha\beta} = \frac{\mathrm{i}}{2J^2}\langle(H_\alpha - J_0 m - \theta)\sigma_\beta\rangle_M$$

which after some algebra translates into the following constraint on the replicon fluctuations

$$M_{31}\delta\boldsymbol{k} + M_{32}\delta\boldsymbol{q} + M_{33}\delta\boldsymbol{r} = 0.$$

The stability of the RS saddle-point against replicon fluctuations is now controlled by a symmetric 2×2 matrix $\overline{\boldsymbol{M}}$:

$$\Psi - \Psi_{\mathrm{RS}} = \sum_{\alpha\beta}\begin{pmatrix}\delta k_{\alpha\beta}\\ \delta q_{\alpha\beta}\end{pmatrix}\overline{\boldsymbol{M}}\begin{pmatrix}\delta k_{\alpha\beta}\\ \delta q_{\alpha\beta}\end{pmatrix} + \mathcal{O}(\delta^3)$$

$$\overline{\boldsymbol{M}} = \begin{pmatrix}1 & 0 & -M_{31}/M_{33}\\ 0 & 1 & -M_{32}/M_{33}\end{pmatrix}\boldsymbol{M}\begin{pmatrix}1 & 0\\ 0 & 1\\ -M_{31}/M_{33} & -M_{32}/M_{33}\end{pmatrix}$$
$$= \begin{pmatrix}M_{11} - M_{13}^2/M_{33} & M_{12} - M_{13}M_{32}/M_{33}\\ M_{12} - M_{13}M_{32}/M_{33} & M_{22} - M_{23}^2/M_{33}\end{pmatrix}. \tag{60}$$

Due to the curvature sign change of the second derivative of Ψ, the analytic continuation to $n\to 0$ of the saddle-point that maximizes Ψ for $n > 1$, will minimize Ψ for $n < 1$. This is emphasized explicitly by the summation over $n(n-1)$ non-trivial terms (all index combinations with $\alpha\neq\beta$) in (58). We can conclude that the AT instability occurs when the largest eigenvalue of the matrix $\overline{\boldsymbol{M}}$ is zero.

4.3. Equilibrium

From our previous result, the confirmation that the general (RSB) thermodynamic equilibrium state is a stationary state of our flow equation (17), it follows that the same must hold within the RS ansatz. We will now show this explicitly, as a non-trivial consistency test (rather than a new result). The previous ansatz (34) translates into

$$\chi(\sigma, H) = \exp[\tfrac{1}{2}\beta\sigma(H+\theta)]. \tag{61}$$

Due to (61) the measure M_{RS} in (46) becomes a Gaussian function of the fields, which enables us to perform the field integrals and work out the RS saddle-point equations (49)–(54). The result is:

$$\mu = \tfrac{1}{2}\beta J_0 m \qquad Q = 0 \qquad R_0 = \tfrac{1}{2}\beta \tag{62}$$

$$m = \int \mathrm{D}z\ \tanh\beta(J_0 m + \theta + Jz\sqrt{q}) \tag{63}$$

$$q = \int \mathrm{D}z\ \tanh^2\beta(J_0 m + \theta + Jz\sqrt{q}) \tag{64}$$

which are the familiar RS equilibrium saddle-point equations, as first obtained in [2].

We now turn to the right-hand side of equation (17). Since $Q = 0$ the original two Gaussian variables (x, y) in (46) are replaced by a single one, z. With (61) and (36) we can simplify the effective measure $M_{\rm RS}$ (47), as in the full RSB case, leading to

$$\langle\!\langle \langle f[\sigma, H]\rangle_\star \langle g[\sigma, H]\rangle_\star \rangle\!\rangle_{xy} \to \int {\rm D}z\, \langle f[\sigma, H]\rangle_\star \langle g[\sigma, H]\rangle_\star$$

$$\langle f[\sigma, H]\rangle_\star = \left[\sum_\sigma \exp\{\beta\sigma[J_0 m + \theta + Jz\sqrt{q}]\}\right.$$
$$\left.\times \int {\rm D}w\, f[\sigma, J_0 m + \theta + Jw\sqrt{1-q} + Jz\sqrt{q} + \beta J^2\sigma(1-q)]\right]$$
$$\times[2\cosh\beta[J_0 m + \theta + Jz\sqrt{q}]]^{-1}.$$

In particular:

$$\langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star = \frac{\exp\{\beta\varsigma h - \frac{1}{2}\beta^2 J^2(1-q) - \frac{1}{2J^2(1-q)}[h - J_0 m - \theta - Jz\sqrt{q}]^2\}}{2J\sqrt{2\pi(1-q)}\cosh\beta[J_0 m + \theta + Jz\sqrt{q}]}. \qquad (65)$$

The dependence of (65) on ς only through a factor $e^{\beta\varsigma h}$ immediately ensures that the first two terms in the diffusion equation (17) cancel. Since this happens even before we carry out the Gaussian average, we may write

$$\langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star = \tfrac{1}{2}[1 + \varsigma\tanh(\beta h)]D(h; z)$$

implying relations like

$$\langle \sigma f(H)\rangle_\star = \langle \tanh(\beta H) f(H)\rangle_\star.$$

The building blocks of (55) thereby become

$$\langle\sigma\rangle_\star = \tanh\beta[J_0 m + \theta + Jz\sqrt{q}]$$
$$\langle H - J_0 m - \theta\rangle_\star = \beta J^2(1-q)\langle\sigma\rangle_\star + Jz\sqrt{q}$$
$$\int {\rm D}z\, \langle\tanh(\beta H)\rangle_\star\langle\sigma\rangle_\star = q$$
$$\int {\rm D}z\, \langle\tanh(\beta H)\rangle_\star\langle H - J_0 m - \theta\rangle_\star = 2\beta J^2 q(1-q)$$
$$\int {\rm D}z\, \langle\tanh(\beta H)(H - J_0 m - \theta)\rangle_\star = \beta J^2(1-q^2).$$

We will also need the following identity, obtained by partial integration over z:

$$\int {\rm D}z\, z\langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star = \frac{\sqrt{q}}{J}D(\varsigma, h)(h - J_0 m - \theta)$$
$$-\beta J\sqrt{q}(1-q)\int {\rm D}z\, \langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star\langle\sigma\rangle_\star. \qquad (66)$$

We now have the necessary tools to analyse with minimum effort the complicated terms in our diffusion equation, given the ansatz (61). The combined flow terms in (17) can be simplified to

$$D(\varsigma, h)(h - J_0 m - \theta)D(\varsigma, h) + \mathcal{A}_{\rm RS}[\varsigma, h; D] = [1 - \langle\tanh^2(\beta H)\rangle_D]$$
$$\left[\frac{1-2q}{(1-q)^2}D(\varsigma, h)(h - J_0 m - \theta) - \beta J^2\varsigma D(\varsigma, h) + \frac{q\beta J^2}{1-q}\right.$$
$$\left.\times\int {\rm D}z\, \langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star\langle\sigma\rangle_\star + \frac{Jq\sqrt{q}}{(1-q)^2}\int {\rm D}z\, z\langle \delta[h - H]\delta_{\varsigma,\sigma}\rangle_\star\right].$$

In order to evaluate the diffusion term in (17) we calculate the field derivative of $D(\varsigma, h)$, using (65):

$$J^2 \frac{\partial}{\partial h} D(\varsigma, h) = \beta J^2 \varsigma D(\varsigma, h) - \frac{h - J_0 m - \theta}{1-q} D(\varsigma, h) + \frac{J\sqrt{q}}{1-q} \int \mathrm{D}z\, z \langle \delta[h-H] \delta_{\varsigma,\sigma} \rangle_\star .$$

The full right-hand side of (17) can now be written as

$$[1 - \langle \tanh^2(\beta H) \rangle_D] \Bigg[\frac{q\beta J^2}{1-q} \int \mathrm{D}z\, \langle \delta[h-H] \delta_{\varsigma,\sigma} \rangle_\star \langle \sigma \rangle_\star - \frac{q}{(1-q)^2} D(\varsigma, h)(h - J_0 m - \theta) + \frac{J\sqrt{q}}{(1-q)^2} \int \mathrm{D}z\, z \langle \delta[h-H] \delta_{\varsigma,\sigma} \rangle_\star \Bigg] = 0$$

(by virtue of the identity (66)). The RS equilibrium state obtained in [2] therefore defines a stationary state of our RS diffusion equation (17), (55).

Finally we turn to the AT instability, which we found to occur when the largest eigenvalue of the matrix $\overline{\boldsymbol{M}}$ (60) is zero. We can use the various identities, derived for the thermal equilibrium state, to simplify the matrix elements of $\overline{\boldsymbol{M}}$ considerably:

$$\overline{\boldsymbol{M}}_{\mathrm{eq}} = \frac{-1}{2(\beta J)^2 \Lambda - 1} \begin{pmatrix} \Lambda & \frac{1}{2}[1 - (\beta J)^2 \Lambda] \\ \frac{1}{2}[1 - (\beta J)^2 \Lambda] & \frac{1}{4}(\beta J)^2 [1 - (\beta J)^2 \Lambda] \end{pmatrix}$$

with

$$\Lambda = \int \mathrm{D}z\, \cosh^{-4} \beta[J_0 m + \theta + Jz\sqrt{q}].$$

The AT instability, as calculated within equilibrium statistical mechanics [9], occurs at $(\beta J)^2 \Lambda = 1$. Substitution of this condition into our expression for $\overline{\boldsymbol{M}}_{\mathrm{eq}}$ immediately leads to the desired result: the two eigenvalues of $\overline{\boldsymbol{M}}_{\mathrm{eq}}$ are $\{-\Lambda, 0\}$, so the two conditions for the AT instability coincide.

5. Comparison with simulations

In order to verify the predictions of our theory we here compare the results of solving numerically the (macroscopic) diffusion equation (17), in which the disorder-generated term $\mathcal{A}$ is calculated within the RS ansatz (55), with the results of performing numerical simulations of the discretized version of the underlying microscopic stochastic dynamics (2), (3). Solving the diffusion equation (17), requires making a discretization not only of time, but also of the joint spin-field distribution, i.e. replace the two continuous functions $D_t(\pm 1, h)$ by two histograms. Furthermore, at each time-step we have to solve the RS saddle-point equations (48)–(54), which involve nested Gaussian integrations. It will be clear that the solution of equation (17) requires a significant computational effort, even within the RS ansatz, which is reflected in the scope of the experiments described in this paper. We restrict ourselves to describing the evolution of the system in zero external field ($\theta = 0$), following initial states with individual spin states chosen independently at random, given a required initial magnetization. Following the various experimental protocols that show spin-glass ageing phenomena, such as relaxation following cooling in a small field, and relaxation with intermittent temperature increases or decreases, we consider to be beyond the scope of this paper.

5.1. Transients

First we study the relaxation of the system on short time-scales. We measure as a function of time the magnetization m, the energy per spin E, and the two distributions $D_t(\pm 1, h)$. Note that the full local field distribution $D_t(h)$ is just the sum $D_t(1, h)+D_t(-1, h)$. The numerical simulations were carried out with systems of $N = 8000$ spins, following randomly drawn initial states. The results of confronting our theory with typical simulation experiments, for relaxations at $T = 0$, are shown in figures 1 and 2, for $J_0 = 0$ (left pictures) and $J_0 = 1$ (right pictures). In figure 1 the top graphs represent the magnetization m and the bottom graphs represent the energy per spin E; for the two initial conditions $m_0 = 0$ and $m_0 = 0.3$. Figure 2 shows the corresponding distributions $D(\sigma, h)$ for one particular choice of initial state ($D_t(1, h)$: upper graph in $t = 0$ window, right graph in $t > 0$ windows; $D_t(-1, h)$: lower graph in $t = 0$ window, left graph in $t > 0$ windows). For $J_0 = 1$ we were unable to calculate the solution of equation (17) up to $t = 6$, due to the critical behaviour of the saddle-point equations (48)–(54). In figures 3 and 4 we show similar relaxation results for $T = 1$. As expected, at higher temperatures the two distributions $D_t(\pm 1, h)$ acquire a shape which becomes more like a Gaussian one, whereas in the low-temperature regime the deviations from a Gaussian shape become important.

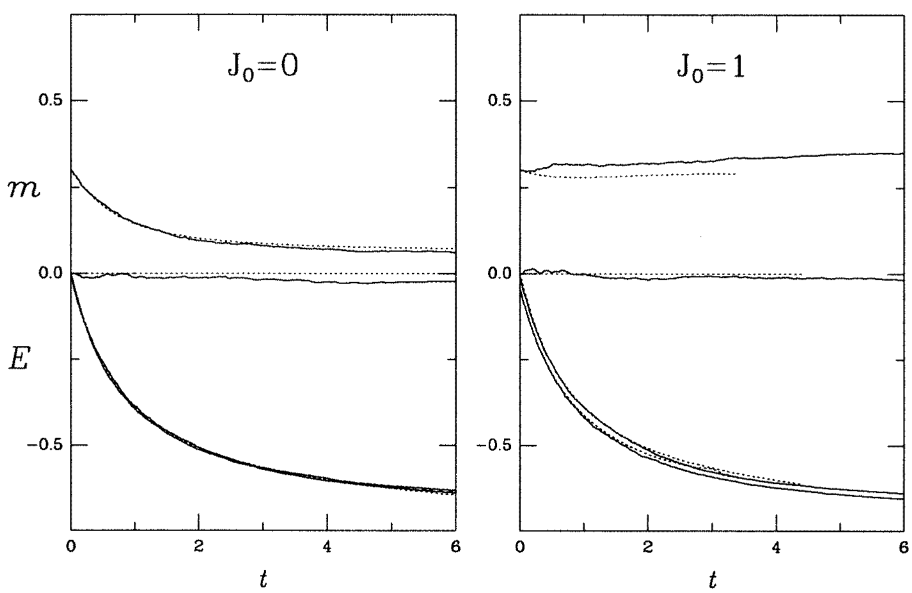

Figure 1. Evolution at $T = 0$ of the magnetization m and the energy per spin E, for $J_0 = 0$ (left) and $J_0 = 1$ (right): full curves, numerical simulations with $N = 8000$; dotted curves, result of solving the RS diffusion equation.

To emphasize the increase in accurateness obtained by the present advanced version of our theory, as opposed to the simple two-parameter theory of [1], we show in figure 5 the simulation data and the predictions of the two versions of our theory (simple as opposed to advanced) corresponding to a relaxation from a random initial state (with $m_0 = 0$), for $T = J_0 = 0$. The failure of the two-parameter theory to account for the typical slowing down of the dynamics appears to have been amended convincingly by choosing as the dynamic object the full distribution $D_t(\sigma, h)$, rather than just the magnetization and the energy per spin. Since the solution of our diffusion equation (17), as depicted in figure 5, is

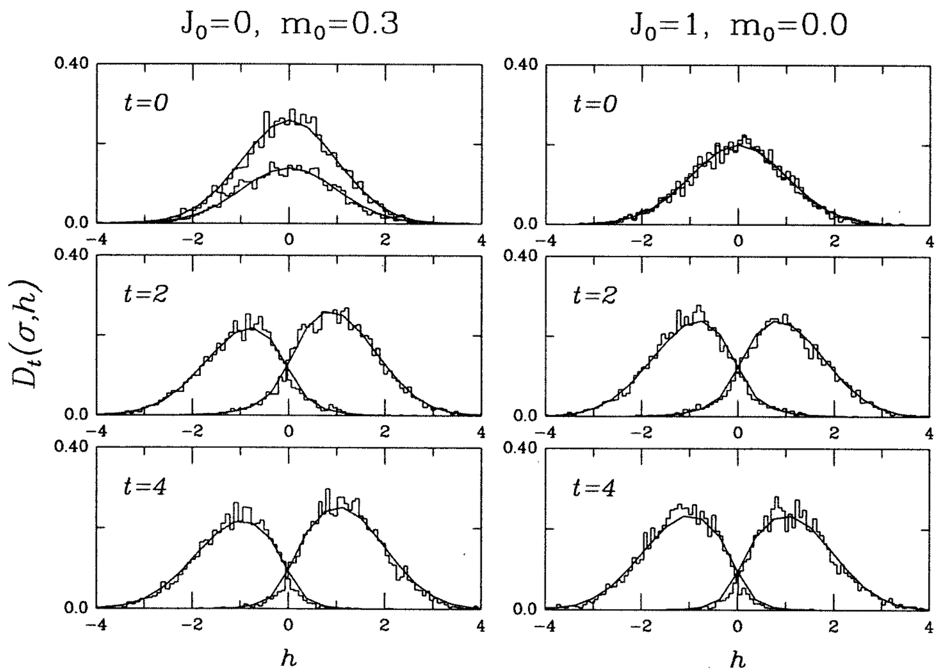

Figure 2. Evolution at $T = 0$ of the two field distributions $D_t(\sigma, h)$, for $J_0 = 0$ (left) and $J_0 = 1$ (right): histograms, numerical simulations with $N = 8000$; full curves, result of solving the RS diffusion equation.

obtained within the RS ansatz, this slowing down of the dynamics is not caused by replica symmetry breaking.

5.2. Relaxation near the spin-glass transition

One way in which we can complement the short-time results presented so far, whilst avoiding having to solve the saddle-point problem (48)–(54) for large times, is to consider the dynamics in the $q = 0$ (paramagnetic) region. This allows us to investigate the relaxation near $J_0 = 0$, $T = 1$ (the critical point which marks the $P \to SG$ transition). In the paramagnetic region the RS saddle-point problem can be solved,

$$q = m = R_1 = 0 \qquad J^2 R_0 = \langle \sigma H \rangle_D \qquad J^4 Q^2 = \langle\langle \langle H \rangle_\star^2 \rangle\rangle_{xy}$$

and the diffusion equation can be expressed entirely in terms of (averages over) the distribution $D_t(\varsigma, h)$ itself. Upon also making use of the invariance of the problem with respect to an overall spin sign change, we can write $D_t(\varsigma, h)$ in terms of a single function, the symmetric part of which is proportional to the local field distribution:

$$D_t(\varsigma, h) = \tfrac{1}{2} F_t(\varsigma h) \qquad \langle f(\sigma H) \rangle_D = \int \mathrm{d}y\, F(y) f(y) = \langle f(y) \rangle_F .$$

In terms of F_t the diffusion equation (17) becomes

$$\frac{\partial}{\partial t} F_t(x) = \tfrac{1}{2}[1 + \tanh(\beta x)] F_t(-x) - \tfrac{1}{2}[1 - \tanh(\beta x)] F_t(x)$$
$$+ J^2 [1 - \langle \tanh(\beta y) \rangle_{F_t}] \frac{\partial^2}{\partial x^2} F_t(x) + \frac{\partial}{\partial x} \{ F_t(x) [x [1 - \langle \tanh(\beta y) \rangle_{F_t}]$$

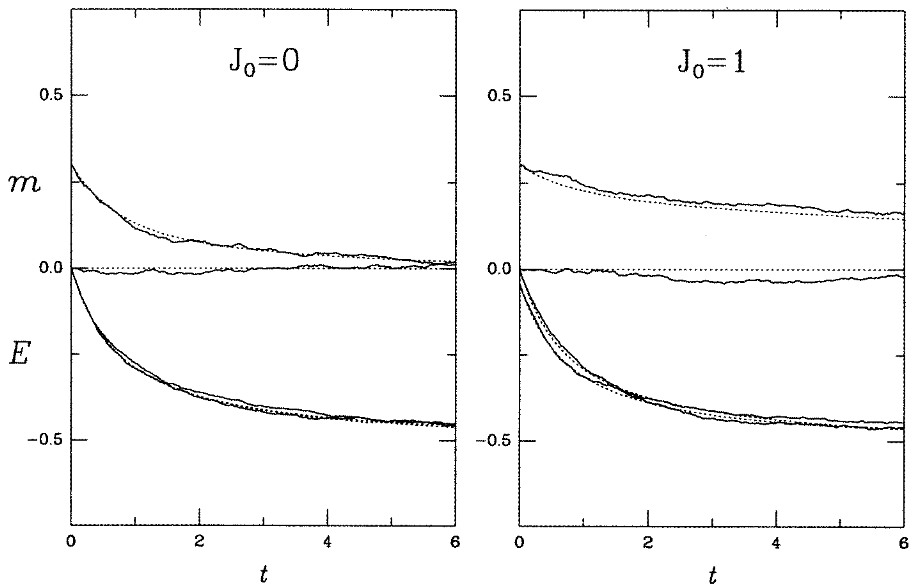

Figure 3. Evolution at $T = 1$ of the magnetization m and the energy per spin E, for $J_0 = 0$ (left) and $J_0 = 1$ (right): full curves, numerical simulations with $N = 8000$; dotted curves, result of solving the RS diffusion equation.

$$+\langle y\rangle_{F_t}\langle\tanh(\beta y)\rangle_{F_t} - \langle y\tanh(\beta y)\rangle_{F_t}]\}. \tag{67}$$

A randomly drawn initial state corresponds to

$$F_0(x) = \frac{1}{J\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2/J^2}.$$

Since (67) is relatively easy to iterate numerically, we can now compare the theoretical predictions with the numerical data over much larger time-scales. In figure 6 we compare the result of solving (67) with numerical simulations, for $t \in [0, 500]$, in terms of the energy per spin $E = -\frac{1}{2}\langle y\rangle_{F_t}$. We observe again a satisfactory agreement between theory and experiment.

6. Discussion

The present paper is the second in a series of papers in which we systematically develop a dynamical replica theory to describe the evolution of macroscopic observables in the Sherrington–Kirkpatrick [2] spin-glass. Our procedure for obtaining closed macroscopic flow equations is based on two assumptions: (i) the flow equations are self-averaging with respect to the realization of the disorder, at any time; and (ii) we may assume equipartitioning of probability in the macroscopic sub-shells of the ensemble. The procedure can be shown to be *exact*, if the set of macroscopic observables to which it is applied indeed obeys closed dynamic equations. The resulting closed flow equations involve a saddle-point problem, to be solved at each instance of time, formulated in the replica language.

In our previous paper [1] the closure procedure was applied to the observables m and E (the magnetization and the energy per spin), resulting in a two-parameter dynamical theory. Here we have shown how the same procedure can be succesfully applied to the

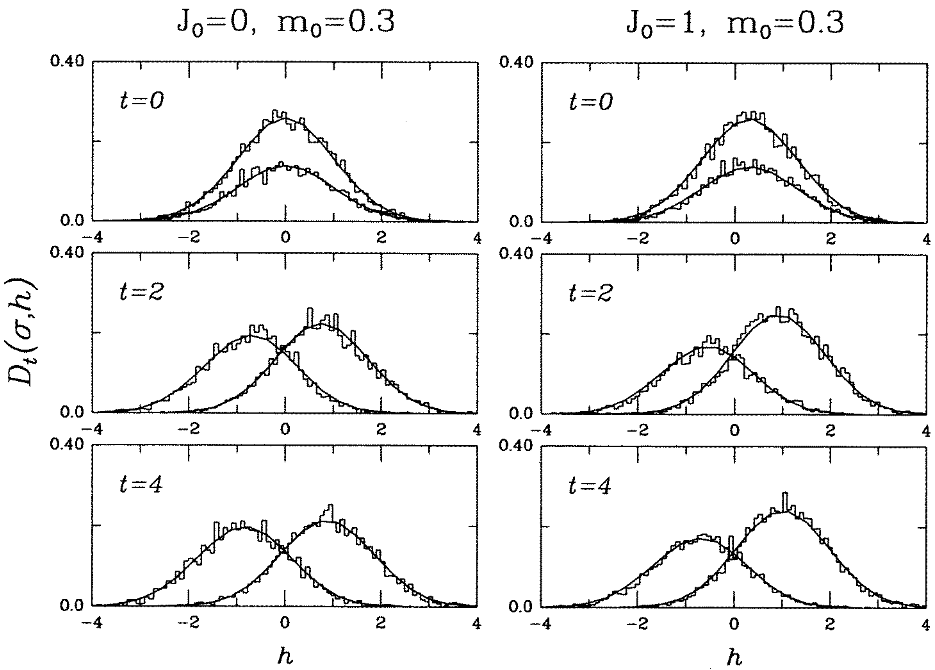

Figure 4. Evolution at $T = 1$ of the two field distribitions $D_t(\sigma, h)$, for $J_0 = 0$ (left) and $J_0 = 1$ (right): histograms, numerical simulations with $N = 8000$; full curves, result of solving the RS diffusion equation.

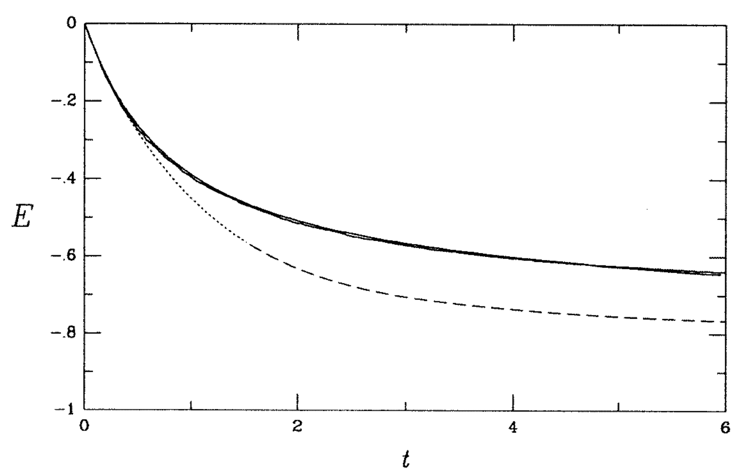

Figure 5. Comparison of simulations ($N = 8000$, full curve), the simple two-parameter theory (RS stable, dotted curve, RS unstable, broken curve) and the advanced theory (full curve), for $T = J_0 = 0$. Note that the two full curves are almost on top of each other at the scale shown.

joint spin-field distribution $D(\varsigma, h)$, resulting in a dynamical theory describing an infinite number of macroscopic order parameters. The present, advanced, version of our theory is again by construction exact for short times, in equilibrium, and in the limit where the disorder is removed. Furthermore, since the joint spin-field field distribution specifies the

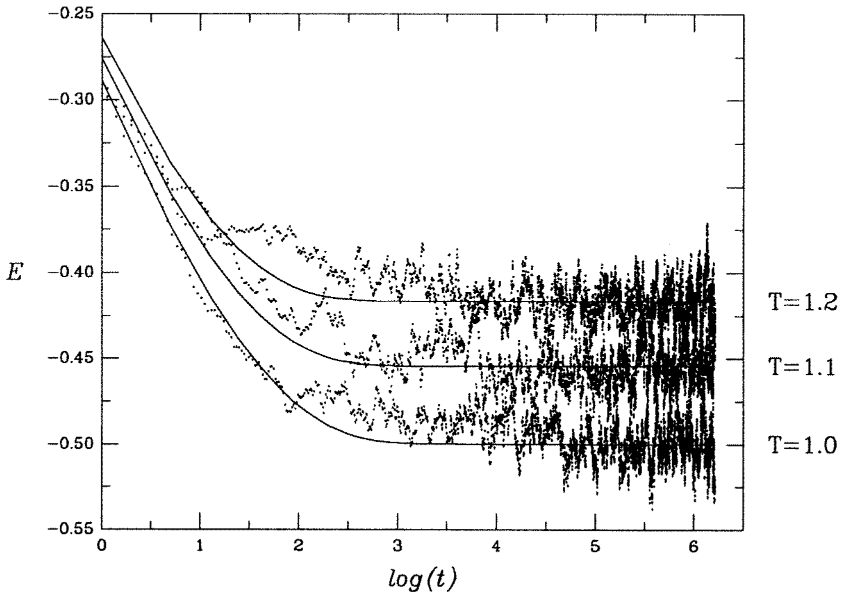

Figure 6. Relaxation of the energy per spin E for $J_0 = \theta = 0$ and $T \in \{1.0, 1.1, 1.2\}$: dots, numerical simulations with $N = 3200$; curves, results of solving the RS diffusion equation.

underlying microscopic states in much more detail than would be the case by specifying only the energy and the magnetization (i.e. more microscopic memory effects are being taken into acount), the equipartitioning assumption has become much weaker. We have restricted our analysis of the saddle-point equations by making the replica-symmetric (RS) ansatz. On the time-scales considered in our simulation experiments, the agreement between advanced RS theory and experiment is quite satisfactory. For example, the slowing down missed by the two-parameter theory is now well accounted for, and the theory describes correctly the relaxation near the spin-glass transition. At this stage we need more efficient numerical procedures in order to extend the time-scales for which we can solve explicitly the analytical macrodynamic equations (17) ff. This would enable us to compare, for instance, with data such as the ones in [10], to investigate the possible existence of stationary states other than the one corresponding to thermal equilibrium, and to see whether the theory can describe the typical ageing phenomena observed in numerical simulations of similar mean-field spin-glass models [11].

A next stage of our programme will be to investigate for the Sherrington–Kirkpatrick spin-glass the effects of replica symmetry breaking on the dynamic equations [6]. Although technically non-trivial, it is a straightforward generalization of the formalism developed so far.

Finally, a relevant question which we have not yet been able to answer is whether our diffusion equation (17), (33) is exact (for infinitely large systems and on finite time-scales). There are several approaches to this problem, each of which we plan to investigate in the near future. The first approach is to apply our formalism to those disordered spin systems for which the dynamics has been solved by other means, like the non-symmetric SK model (in which each of the bonds is drawn independently and asymmetrically at random [12, 13]; preliminary results of this study can be found in [14]), a toy model used in analysing the shortcomings of the previous two-parameter approach [15], or the spherical spin-glass [16].

By definition, however, such exercises would not yet prove exactness in the case of the SK spin-glass. The second approach would be to try to derive a diffusion equation for the joint spin-field distribution, starting from the equations for correlation and response functions, as obtained from the path-integral formalism [5]. The latter approach involves (rather complicated) closed equations for two functions $C(t, t')$ and $R(t, t')$, with two real-valued arguments each (two times). The present formalism also involves two functions $D_t(1, h)$ and $D_t(-1, h)$, with two real-valued arguments each (one time and one field). It is therefore quite imaginable that both formalisms constitute exact discriptions of the dynamics of the SK model.

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References

[1] Coolen A C C and Sherrington D 1994 *J. Phys. A: Math. Gen.* **27** 7687
[2] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
[3] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** 1101; 1983 *Phys. Rev. Lett.* **50** 1946
[4] Zinn-Justin J 1993 *Quantum Field Theory and Critical Phenomena* (Oxford: Oxford University Press)
[5] Sommers H J 1987 *Phys. Rev. Lett.* **58** 1268
[6] Coolen A C C, Laughton S N and Sherrington D 1995 in preparation
[7] Horner H 1990 *Z. Phys.* B **78** 27
[8] Thomsen M, Thorpe M F, Choy T C, Sherrington D and Sommers H J 1986 *Phys. Rev.* B **33** 1931
[9] de Almeida J R L and Thouless D J 1978 *J. Physique* A **11** 983
[10] Kinzel W 1986 *Phys. Rev.* B **33** 5086
[11] Cugliandolo L F, Kurchan J and Ritort F 1994 *Phys. Rev.* B **49** 6331
[12] Crisanti A and Sompolinsky H 1988 *Phys. Rev.* A **37** 4865
[13] Rieger H, Schreckenberg M and Zittartz J 1989 *Z. Phys.* B **74** 527
[14] Coolen A C C, Laughton S N and Sherrington D 1995 *Phys. Rev.* B to be published
[15] Coolen A C C and Franz S 1994 *J. Phys. A: Math. Gen.* **27** 6947
[16] Cugliandolo L F and Dean D S 1995 *J. Phys. A: Math. Gen.* **28** 4213